UNIVERSITY OF
SOUTH CAROLINA

CSCE 590 INTRODUCTION TO IMAGE PROCESSING

Statistics of Images

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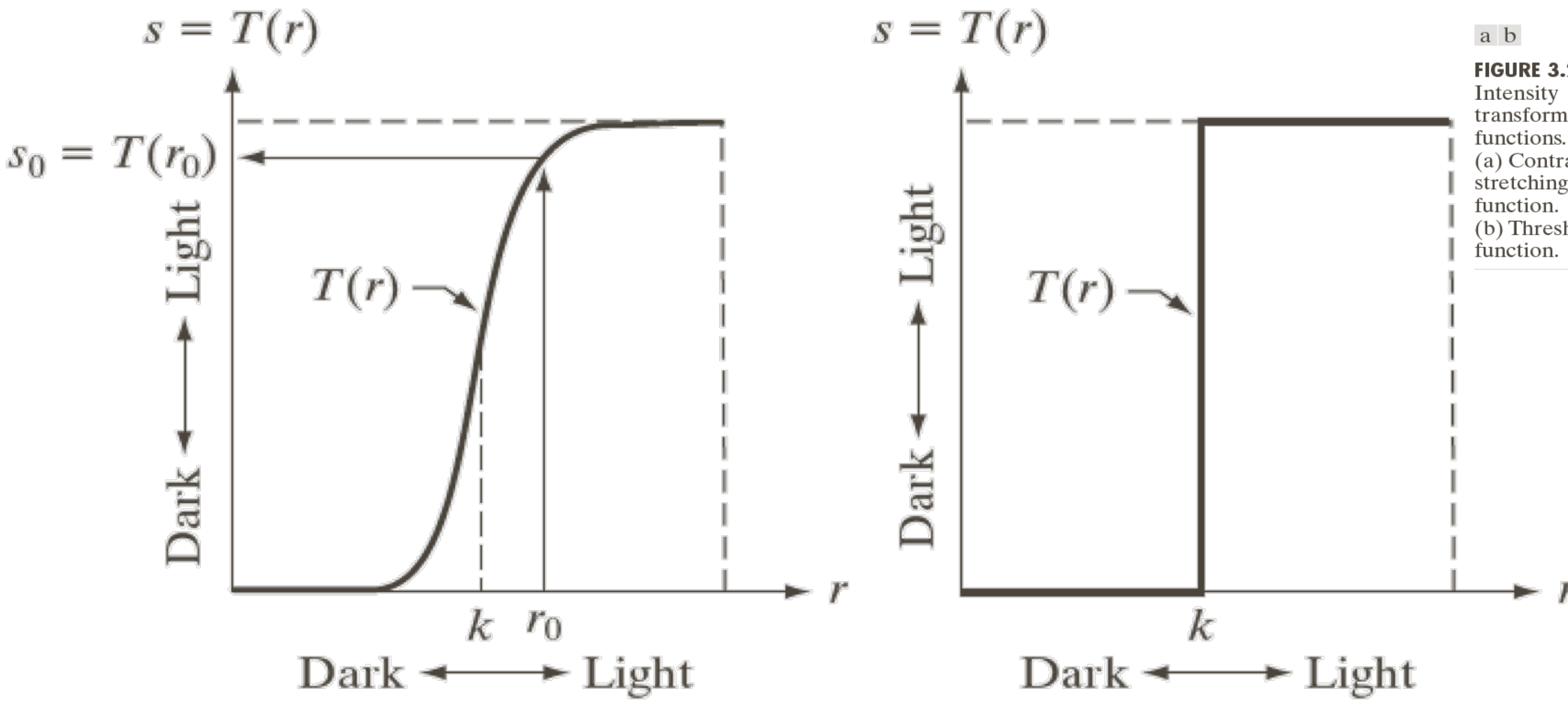
Some Basic Intensity Transformation Functions

- Thresholding – Logistic function
- Log transformation
- Power-law (Gamma correction)
- Piecewise-linear transformation
- Histogram processing

1x1 Neighborhood - Intensity Transformation - Image Enhancement

Contrast stretch



a b

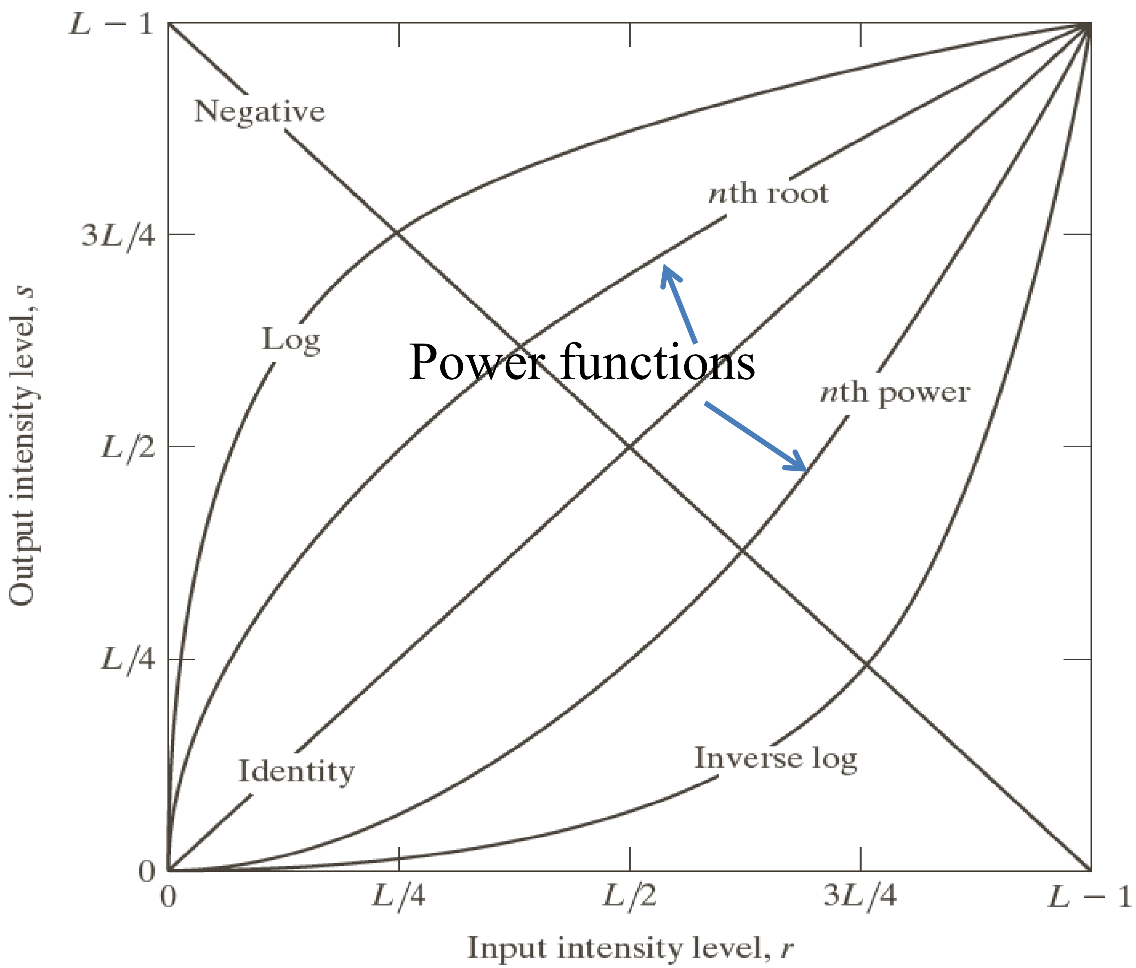
FIGURE 3.2 Intensity transformation functions. (a) Contrast-stretching function. (b) Thresholding function.

Soft thresholding (logistic function)

Hard thresholding (step function)

$$\sim s = \frac{1}{1 + e^{r}}$$

Basic Intensity Transformation Functions



Log function:

$$s = c\log(1 + r) \quad r \geq 0$$

Inverse log function:

$$s = c\log^{-1}(r)$$

FIGURE 3.3 Some basic intensity transformation functions. All curves were scaled to fit in the range shown.

Basic Intensity Transformation Functions

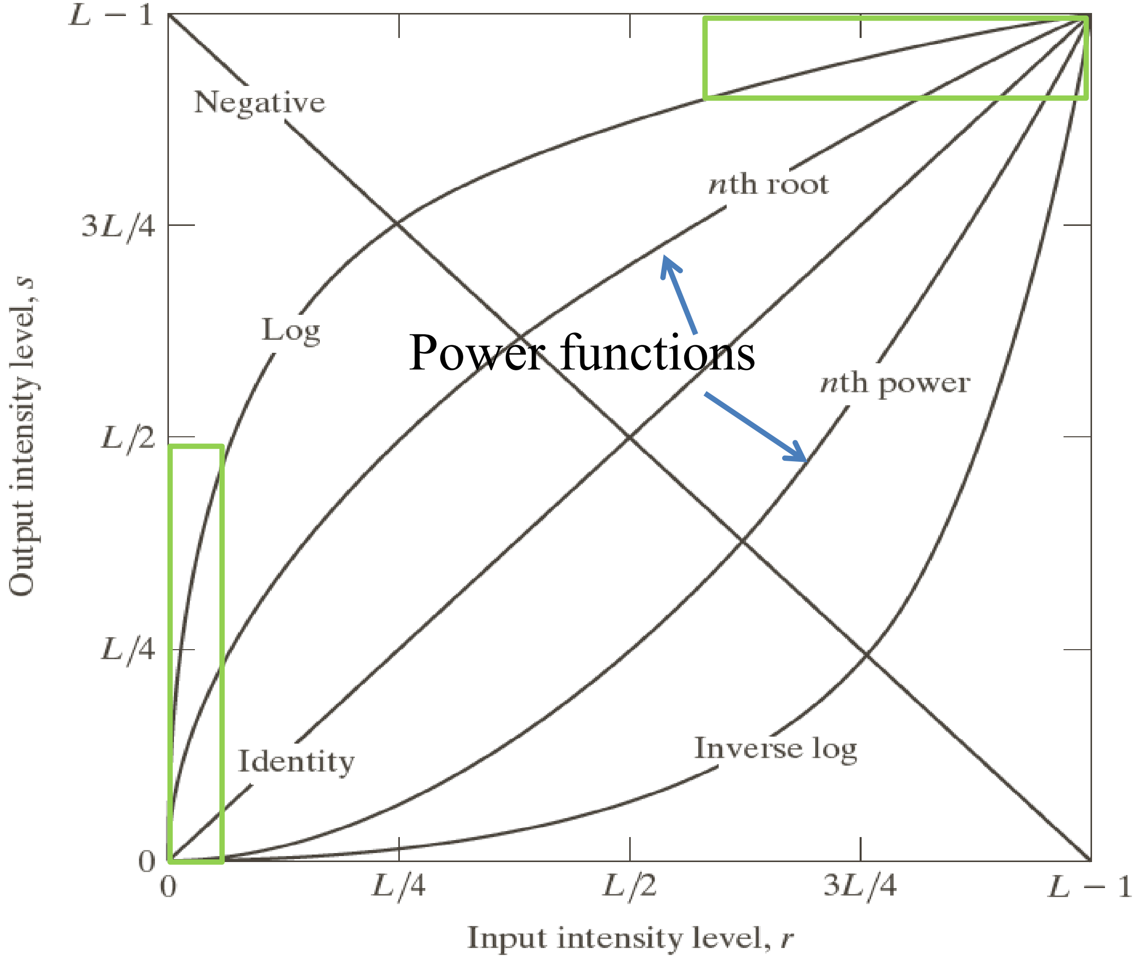


FIGURE 3.3 Some basic intensity transformation functions. All curves were scaled to fit in the range shown.

Log function:

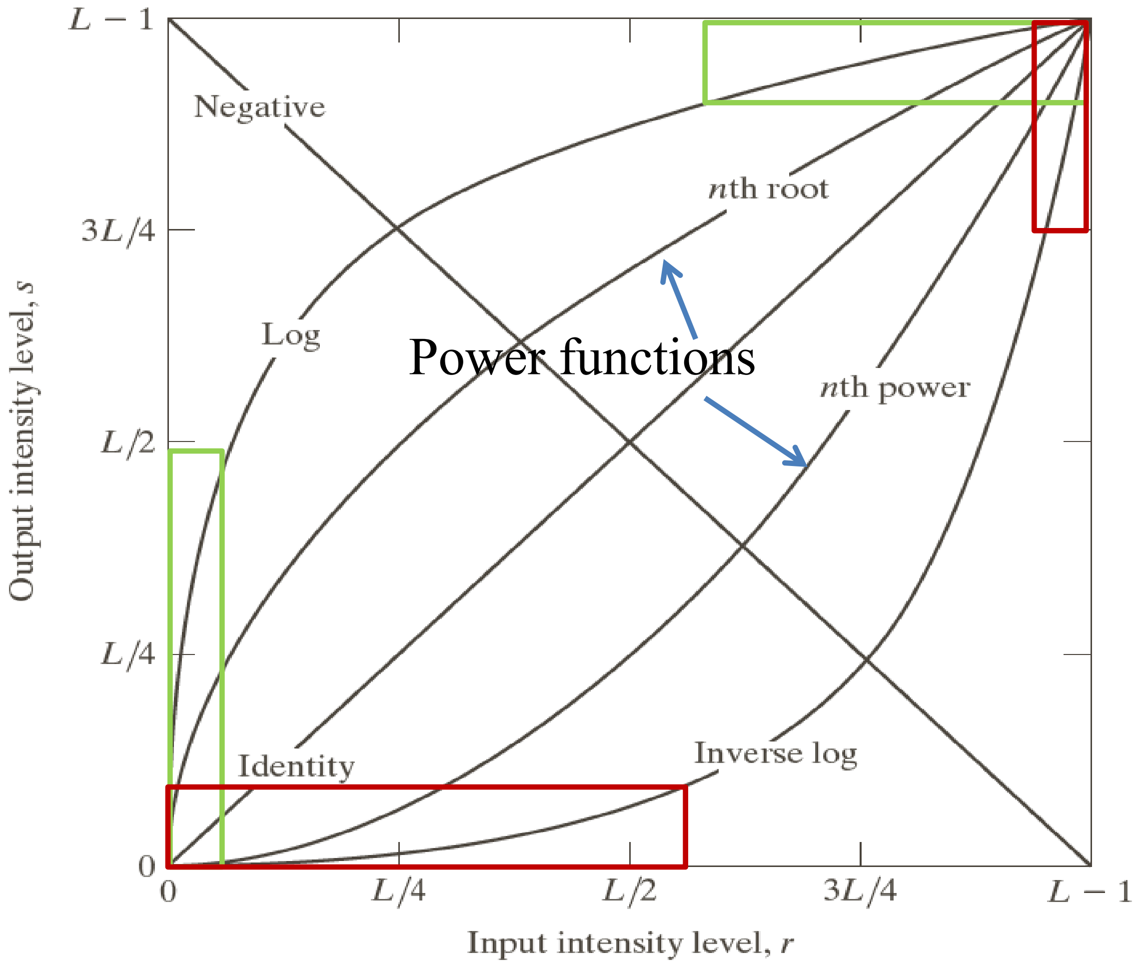
$$s = c\log(1+r) \quad r \geq 0$$

Stretch low intensity levels
Compress high intensity levels

Inverse log function:

$$s = c\log^{-1}(r)$$

Basic Intensity Transformation Functions



Log function:

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Stretch low intensity levels
Compress high intensity levels

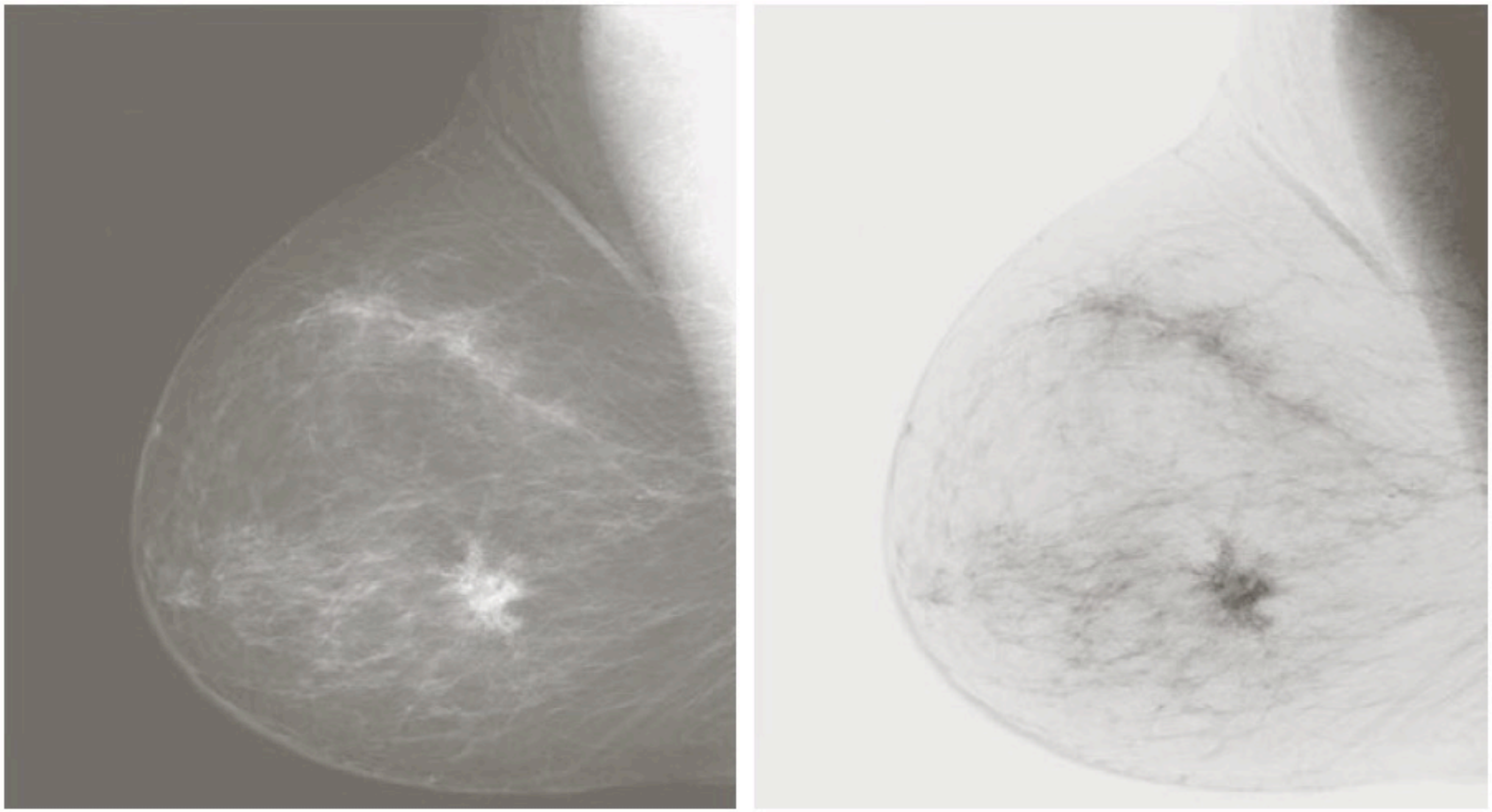
Inverse log function:

$$s = c\log^{-1}(r)$$

Stretch high intensity levels
Compress low intensity levels

FIGURE 3.3 Some basic intensity transformation functions. All curves were scaled to fit in the range shown.

Some Basic Intensity Transformation Functions



a b

FIGURE 3.4 (a) Original digital mammogram. (b) Negative image obtained using the negative transformation in Eq. (3.2-1). (Courtesy of G.E. Medical Systems.)

Image Negative: $s = L - 1 - r$

Log Transformations: $s=c\log(1+r)$

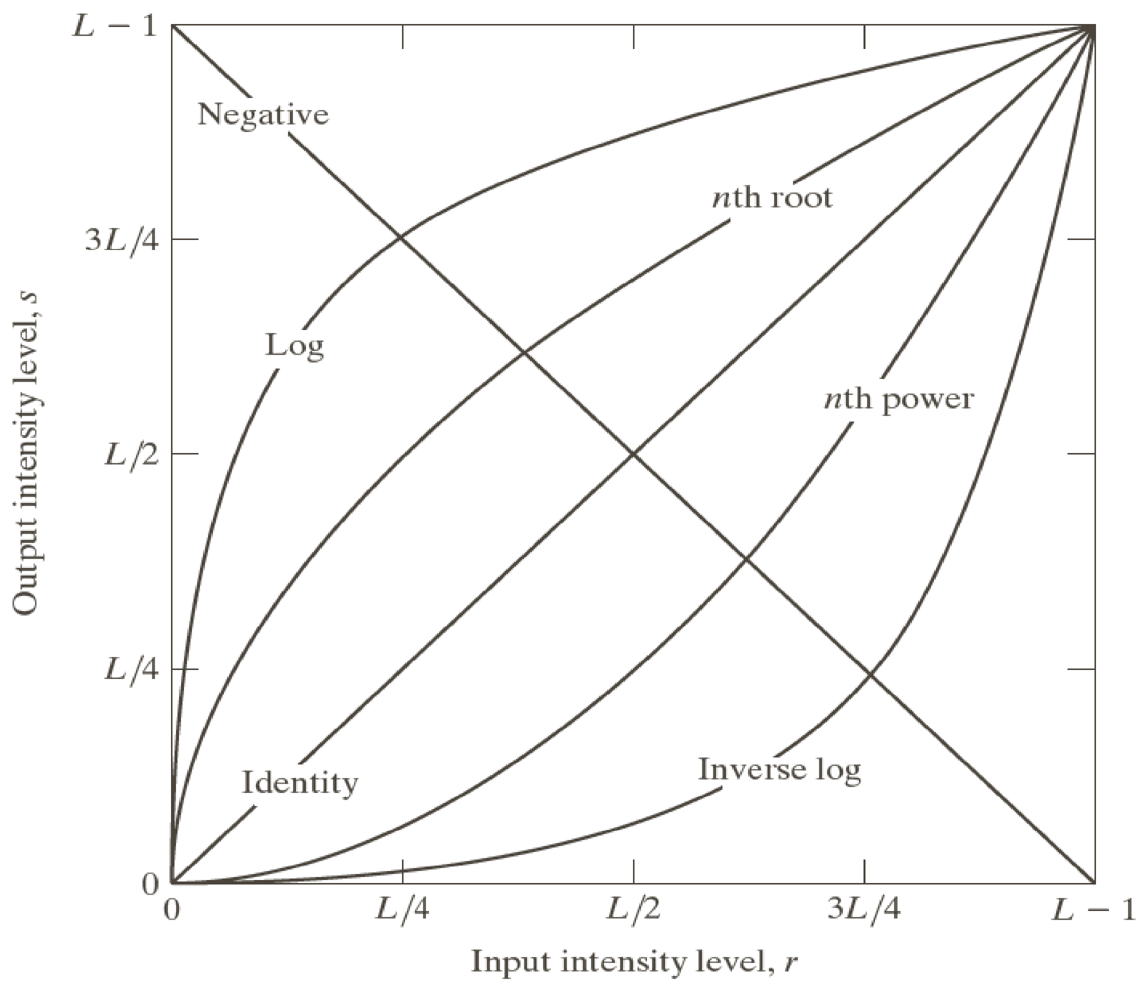


FIGURE 3.3 Some basic intensity transformation functions. All curves were scaled to fit in the range shown.

Power-Law (Gamma) Transformations

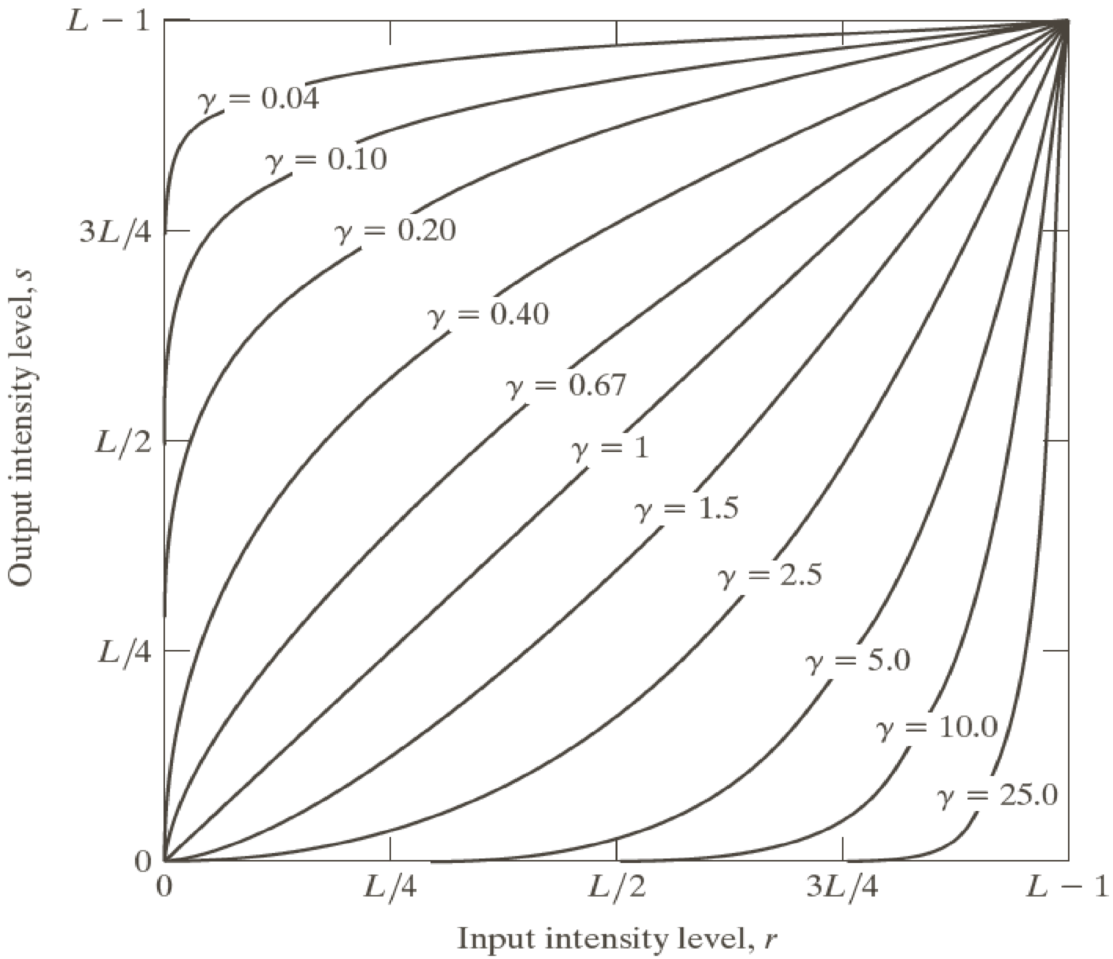


FIGURE 3.6 Plots of the equation $s = cr^{\gamma}$ for various values of γ ($c = 1$ in all cases). All curves were scaled to fit in the range shown.

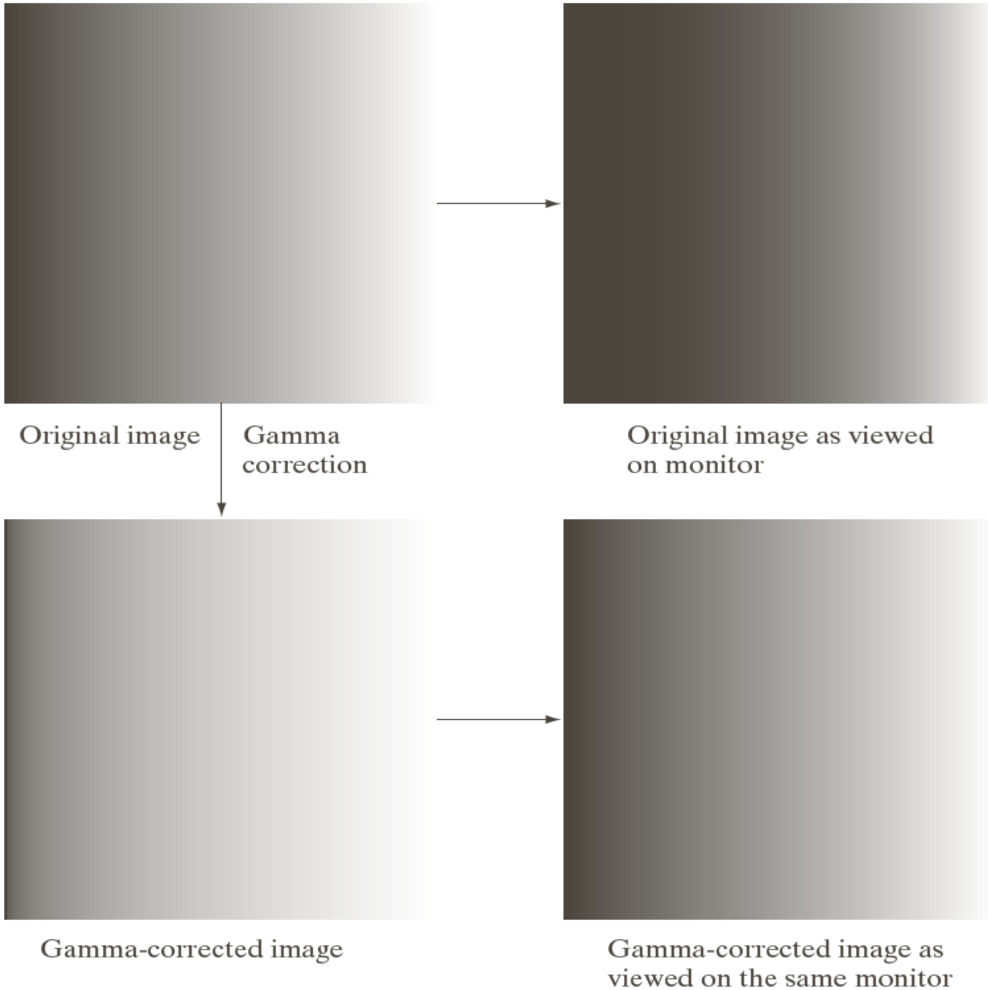
$$s = cr^{\gamma}$$

- More versatile than log transformation
- Performed by a lookup table

LookUp Table Operations

- Look Up Table: LUT[i]=c*i^gamma;
- NI[i,j]=LUT[I[i,j]];

Power-Law (Gamma) Transformations



Monitors have an intensity-to-voltage response with a power function

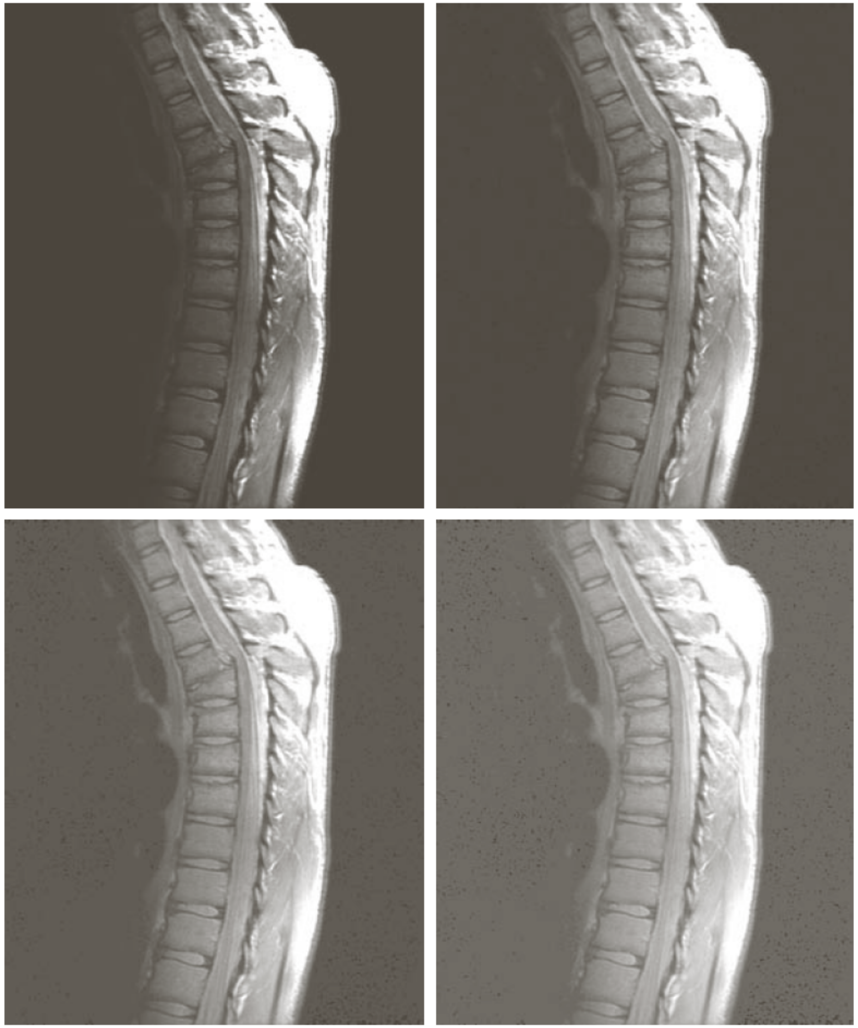
$$s = r^{1/2.5}$$

a	b
c	d

FIGURE 3.7
(a) Intensity ramp image. (b) Image as viewed on a simulated monitor with a gamma of 2.5. (c) Gamma-corrected image. (d) Corrected image as viewed on the same monitor. Compare (d) and (a).

Image Enhancement Using Gamma Correction

Power-Law (Gamma) Transformations for Contrast Manipulation



a b
c d

FIGURE 3.8 (a) Magnetic resonance image (MRI) of a fractured human spine. (b)–(d) Results of applying the transformation in Eq. (3.2-3) with $c = 1$ and $\gamma = 0.6, 0.4,$ and 0.3, respectively. (Original image courtesy of Dr. David R. Pickens, Department of Radiology and Radiological Sciences, Vanderbilt University Medical Center.)

Washed-out appearance caused by a small gamma value

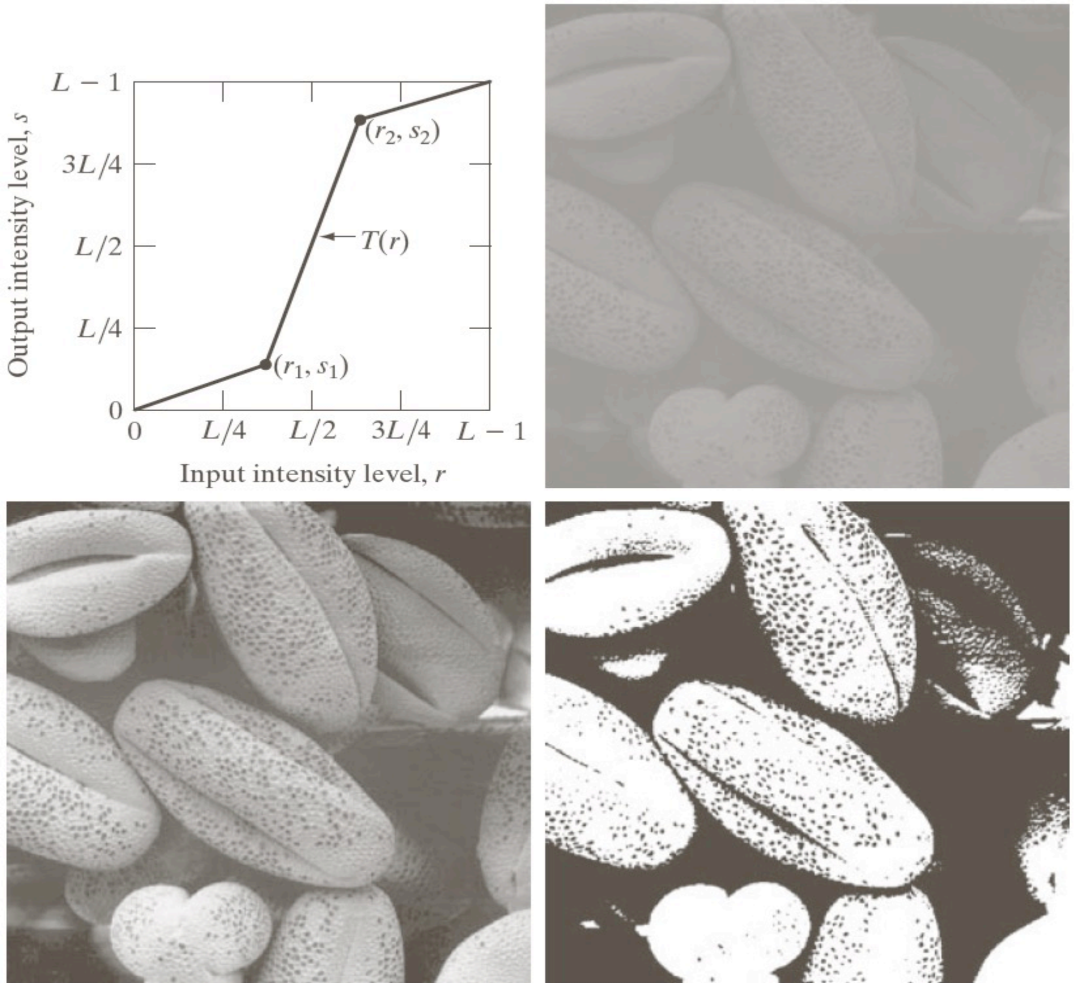
Power-Law (Gamma) Transformations for Contrast Manipulation

a b
c d

FIGURE 3.9 (a) Aerial image. (b)–(d) Results of applying the transformation in Eq. (3.2-3) with $c = 1$ and $\gamma = 3.0, 4.0$, and 5.0, respectively. (Original image for this example courtesy of NASA.)

Washed-out appearance was reduced by a large gamma value

Piecewise-Linear Transformation Functions: Contrast Stretching



a b
c d

FIGURE 3.10 Contrast stretching. (a) Form of transformation function. (b) A low-contrast image. (c) Result of contrast stretching. (d) Result of thresholding. (Original image courtesy of Dr. Roger Heady, Research School of Biological Sciences, Australian National University, Canberra, Australia.)

Piecewise-Linear Transformation Functions: Intensity-Level Slicing

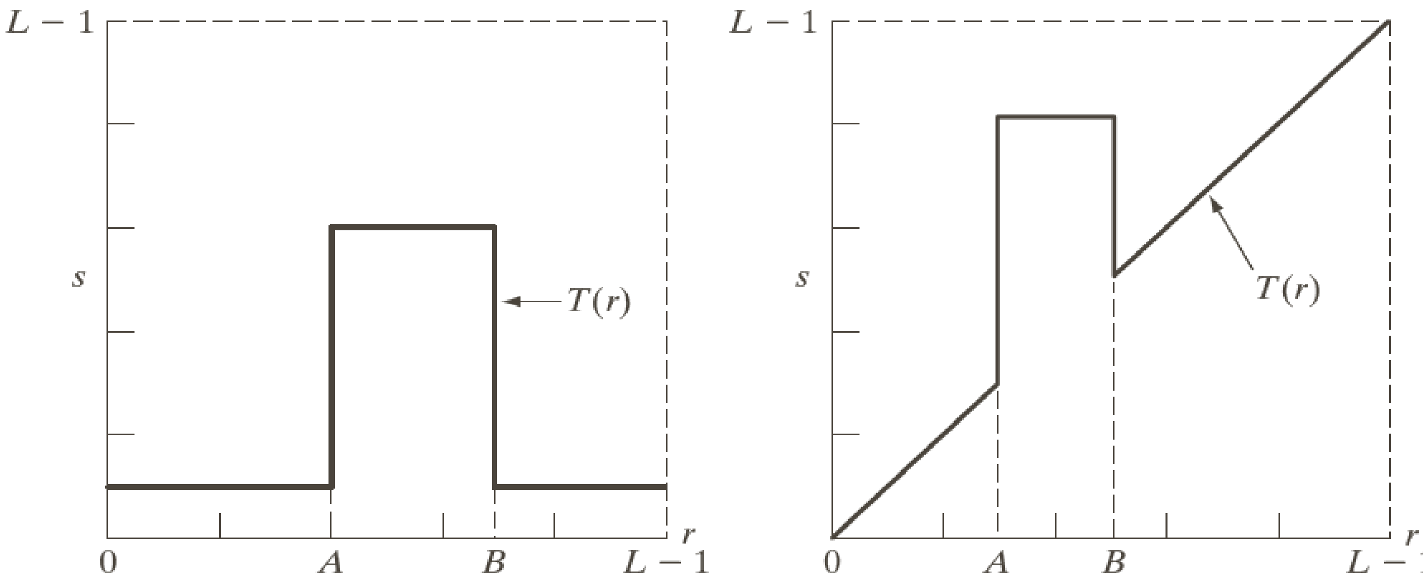
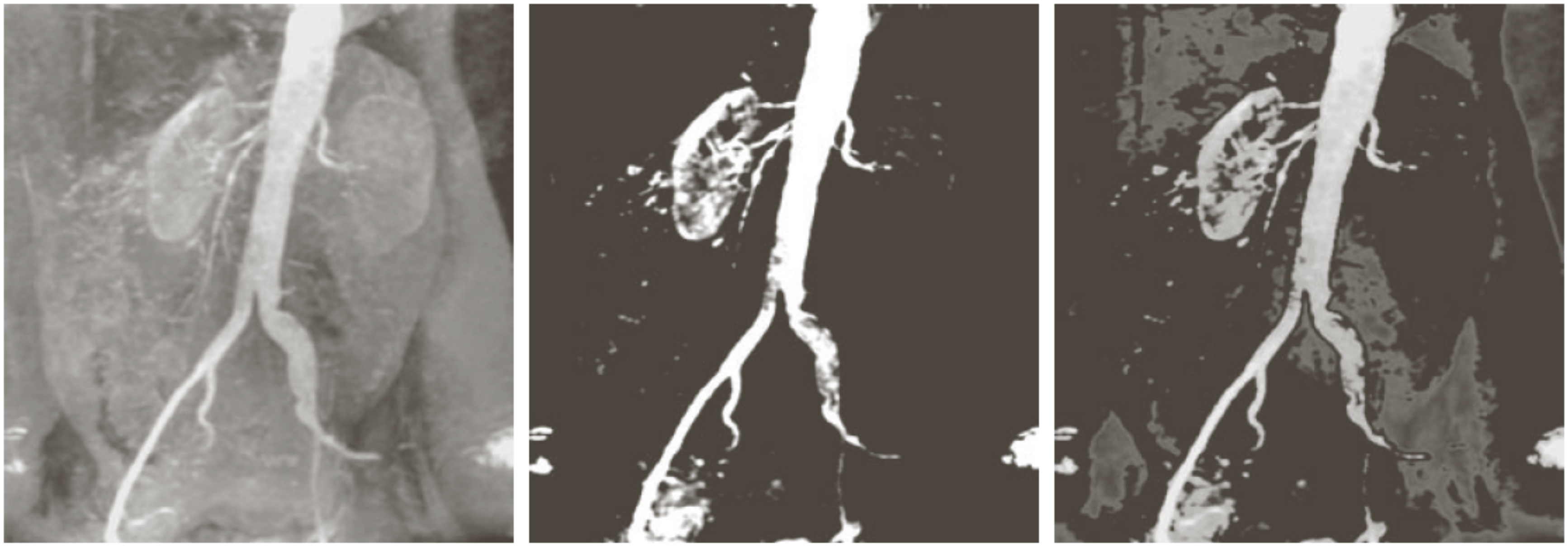


FIGURE 3.11 (a) This transformation highlights intensity range [A, B] and reduces all other intensities to a lower level. (b) This transformation highlights range [A, B] and preserves all other intensity levels.

An Example of Intensity-Level Slicing



a b c

FIGURE 3.12 (a) Aortic angiogram. (b) Result of using a slicing transformation of the type illustrated in Fig. 3.11(a), with the range of intensities of interest selected in the upper end of the gray scale. (c) Result of using the transformation in Fig. 3.11(b), with the selected area set to black, so that grays in the area of the blood vessels and kidneys were preserved. (Original image courtesy of Dr. Thomas R. Gest, University of Michigan Medical School.)

Piecewise-Linear Transformation Functions: Bit-Plane Slicing

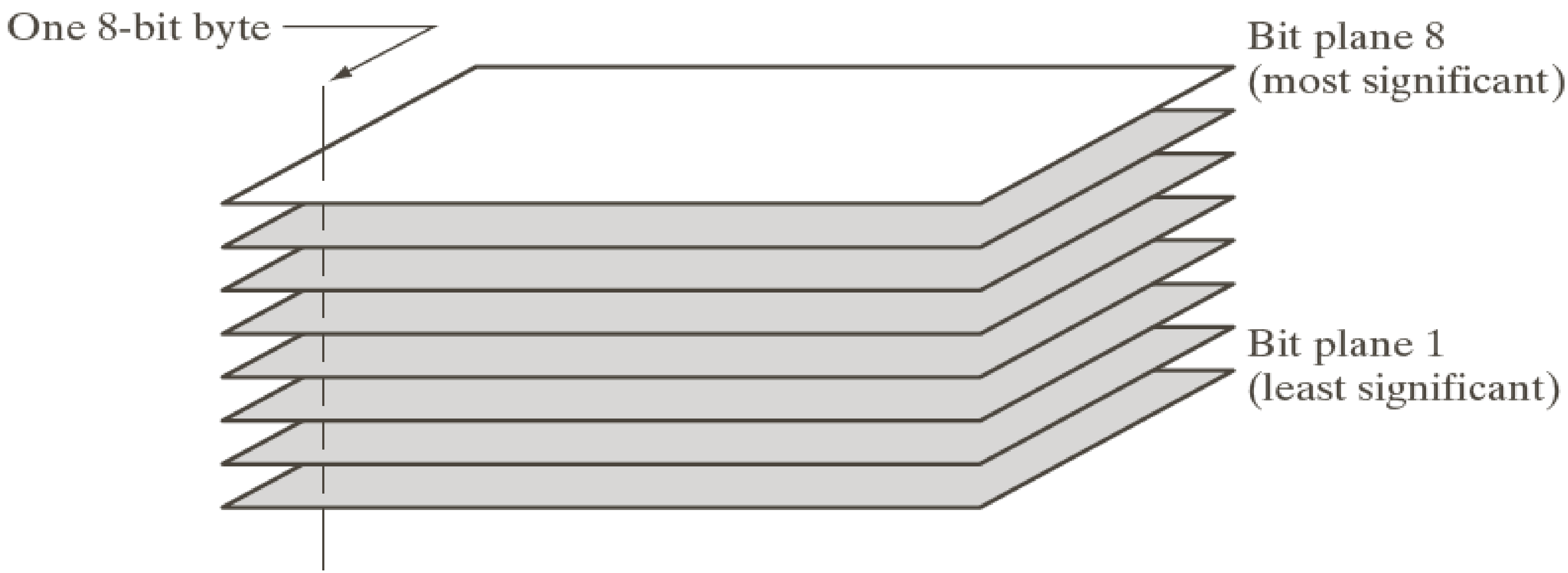


FIGURE 3.13
Bit-plane representation of an 8-bit image.

An Example

a b c
d e f
g h i

FIGURE 3.14 (a) An 8-bit gray-scale image of size 500 × 1192 pixels. (b) through (i) Bit planes 1 through 8, with bit plane 1 corresponding to the least significant bit. Each bit plane is a binary image.

Use for Image Compression

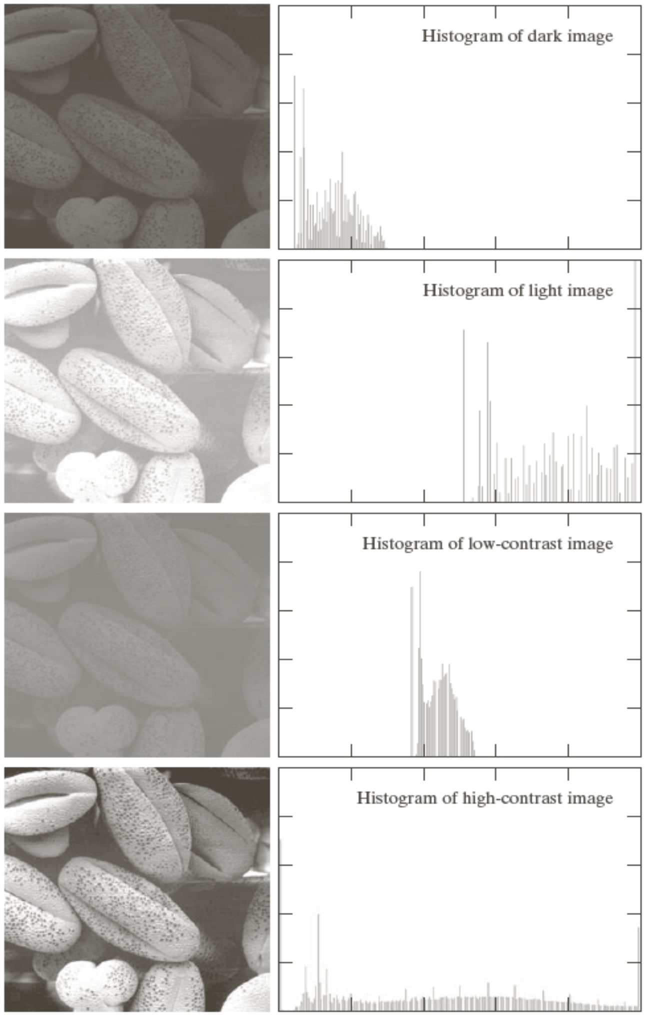
original

a b c

FIGURE 3.15 Images reconstructed using (a) bit planes 8 and 7; (b) bit planes 8, 7, and 6; and (c) bit planes 8, 7, 6, and 5. Compare (c) with Fig. 3.14(a).

Less bit planes are sufficient to obtain an acceptable details, while require half of the storage

Histogram Processing



Histogram

$$h(r_k) = n_k$$

Normalized histogram

$$p(r_k) = n_k / MN$$

$$\sum_{k=0}^{255} p(r_k) = 1$$

FIGURE 3.16 Four basic image types: dark, light, low contrast, high contrast, and their corresponding histograms.

Probabilities Overview

- **What are the chances of rolling a six with a single dice**: P(x=6)
- **Flip a coin**: p(heads)
- **Draw a card from a deck**: p(ace of diamonds)?
- p(x) in [0,1]
- p(x) = times x happen/total number of experiments
- Often express as a percentage
- P(A|B) probability of A happen if B happened
- Discrete P(A) and Continuous P(x)
- P(A or B)=P(A)+P(B)-P(A and B)
- Cumulative Distribution Functions P(x<a)

Probabilities Overview

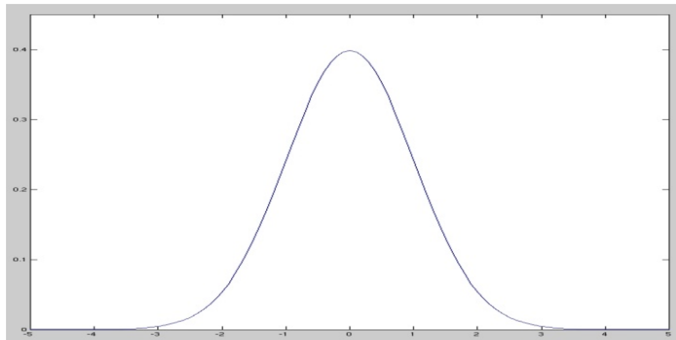
- Conditional probabilities p(a and b) = p(a | b) p(b)
- Bayes theorem relates conditional probabilities

$$p(\ a \mid b\ ) = \frac{p(\ b \mid a\ )p(a)}{p(\ b\ )}$$

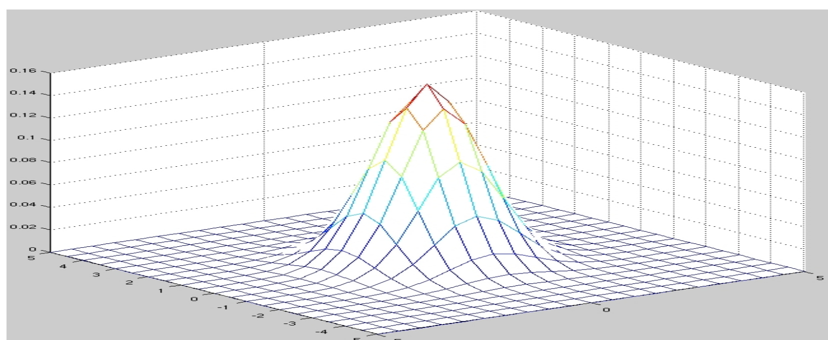
- Expected value, mean: $\mu \equiv \mathrm{E}[X] = \int_{-\infty}^{+\infty} x f(x) dx$
- Variance: $\mathrm{Var}(X) = \mathrm{E}\left[(X-\mu)^2\right].$
- Standard Deviation: $\sigma \equiv \sqrt{\mathrm{E}\left[(X-\mu)^2\right]} = \sqrt{\int_{-\infty}^{+\infty} (x-\mu)^2 f(x) dx},$

Probabilities Overview

- A 1-d Gaussian distribution is given by: $$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



- An n-d Gaussian distribution is given by: $$P(x) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$$



Transformation Function

$$s = T(r) \quad 0 \le r \le L-1$$

A valid transformation function must satisfy two conditions:

$(a)\ T(r)$ is monotonically increasing, i.e., $T(r_1) \ge T(r_2)$ if $r_1 > r_2$

$(b)\ 0 \le T(r) \le L-1$

$(a')\ T(r)$ is strictly monotonic : one - to - one mapping $r = T'(s)$

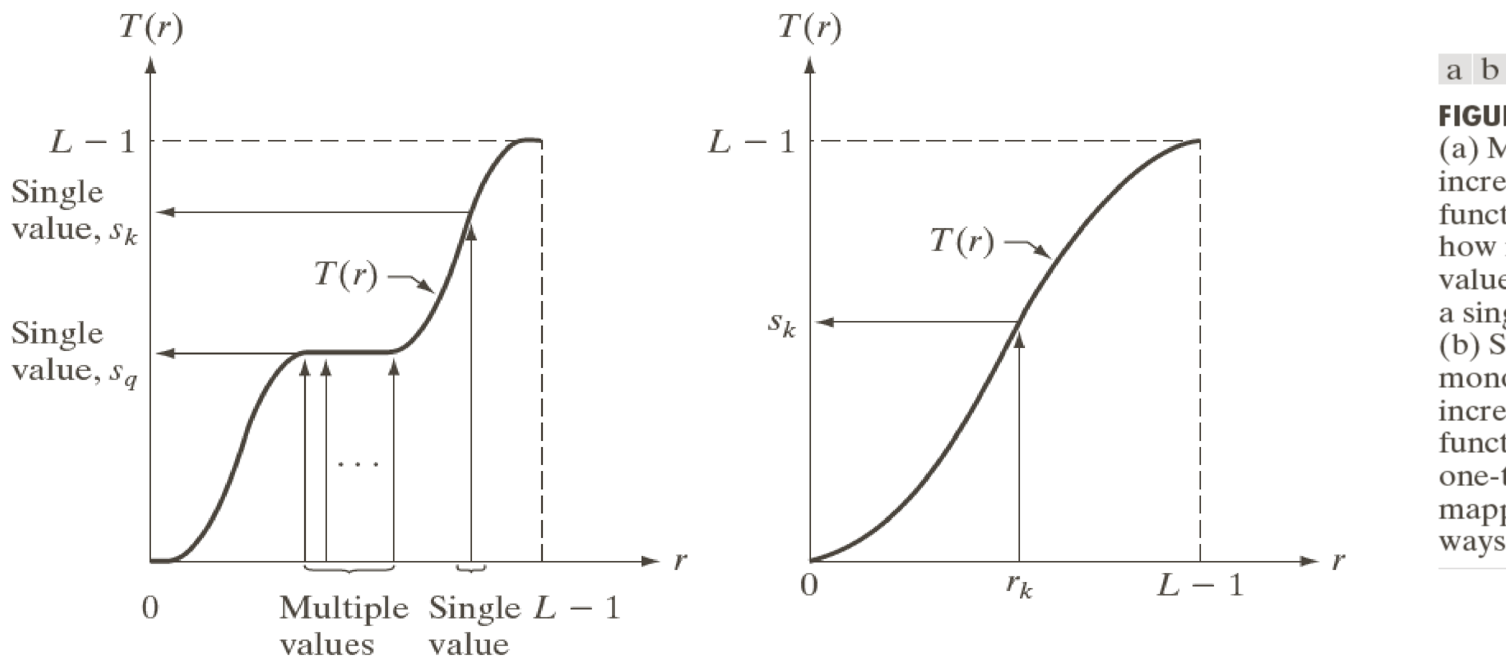


FIGURE 3.17 (a) Monotonically increasing function, showing how multiple values can map to a single value. (b) Strictly monotonically increasing function. This is a one-to-one mapping, both ways.

Histogram Processing

If $T(r)$ is continuous and differentiable.

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

Probability density function of output intensity value

Histogram Equalization

A special transformation function

$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

Cumulative distribution function of r

Is it a valid transformation function?

Yes.

$(a)\ T(r)$ is monotonically increasing, i.e., $\mathrm{T}(\mathrm{r}_1) \geq \mathrm{T}(\mathrm{r}_2)$ if $\mathrm{r}_1 > \mathrm{r}_2$

$(b)\ 0 \leq T(r) \leq L-1$

Cumulative Function

Value	Frequency	Cumulative	T(r)
0	45	45	(45/280)*7
1	64	109	(109/280)*7
2	3	112	(112/280)*7
3	67	179	(179/280)*7
4	12	191	(191/280)*7
5	3	194	(194/280)*7
6	8	202	(202/280)*7
7	78	280	(280/280)*7

Histogram Equalization

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|, \quad s = T(r) = (L-1)\int_0^r p_r(w)dw$$

$$\Rightarrow \quad p_s(s) = \frac{1}{L-1}$$

How to prove it?

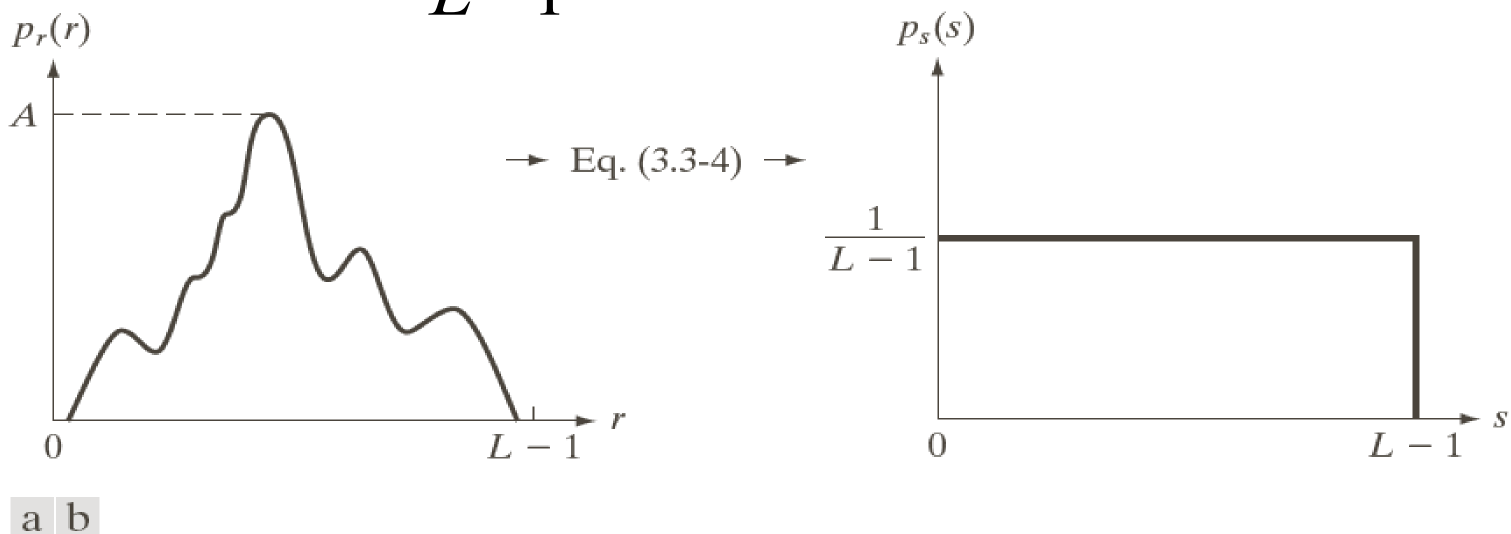


FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

$ds/dr = d(T(r))/dr = (L-1)\, d(S^r p(w)dw)/dr$

$= (L-1)p(r)$

Histogram Equalization – Discrete Case

$$p_r(r_k) = n_k / MN, k = 0,1,2,\ldots,L-1$$

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) = \frac{L-1}{MN}\sum_{j=0}^{k} n_j$$

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

TABLE 3.1 Intensity distribution and histogram values for a 3-bit, 64×64 digital image.

Histogram Equalization – Discrete Case

$$p_r(r_k) = n_k / MN, k = 0,1,2,\ldots,L-1$$

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) = \frac{L-1}{MN}\sum_{j=0}^{k} n_j$$

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
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$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

$p_r(r_k)=n_k/4096$, k=01,2,…,7

$s_k=T(r_k)=(7/4096)*Sum_k(n_k)$

r_k	n_k	$p_r(r_k)$	$Sum_k(n_k)$	$T(r_k)$ calculation	$T(r_k)$ real	$S_k=T(r_k)$
0	790	0.19	790	(7/4096)*790	1.35	1
1	1023	0.25	1813	(7/4096)*1813	3.09	3
2	850	0.21	2663	(7/4096)*2663	4.55	4
3	656	0.16	3319	(7/4096)*3319	5.67	5
4	329	0.08	3648	(7/4096)*3648	6.23	6
5	245	0.06	3893	(7/4096)*3893	6.65	6
6	122	0.03	4015	(7/4096)*4015	6.86	6
7	81	0.02	4096	(7/4096)*4096	7	7

Histogram Equalization – Discrete Case

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

TABLE 3.1 Intensity distribution and histogram values for a 3-bit, 64 × 64 digital image.

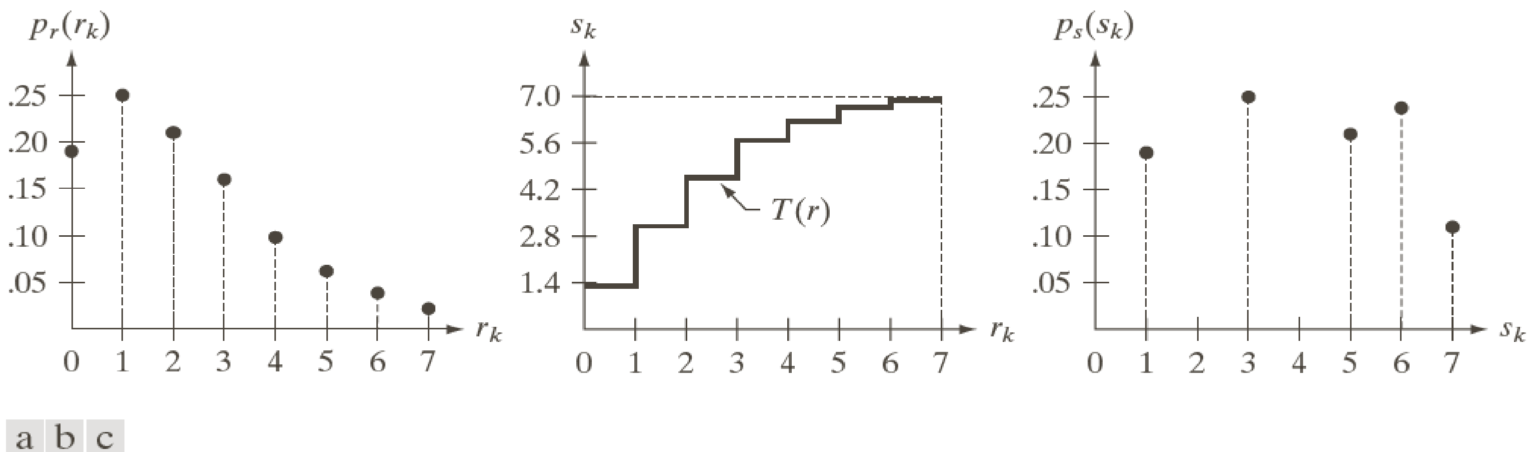
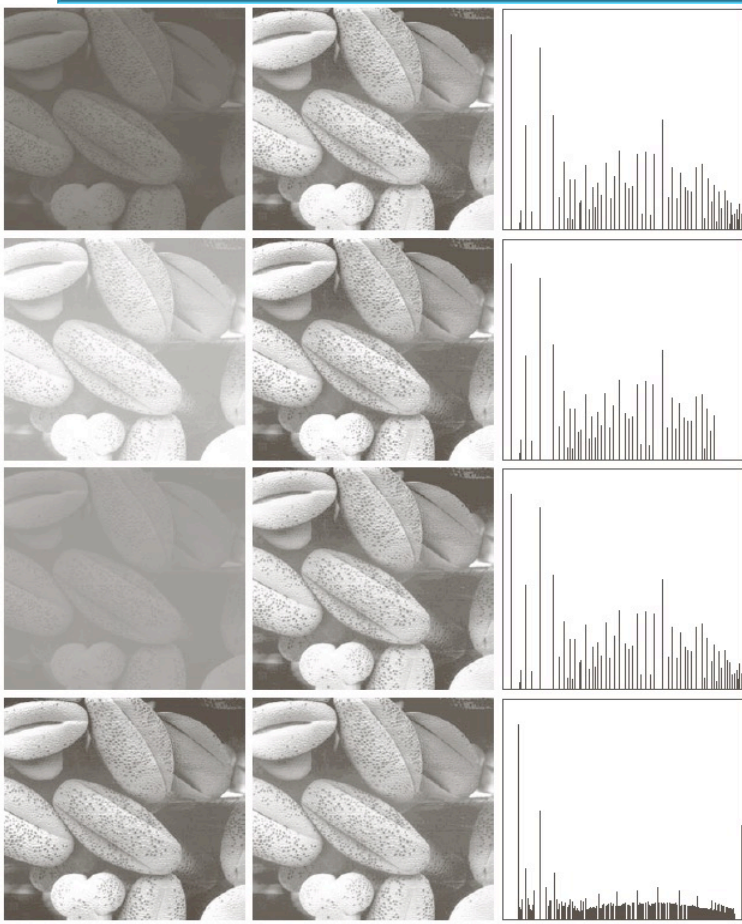


FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

Histogram equalization cannot result in a uniform histogram.

Examples



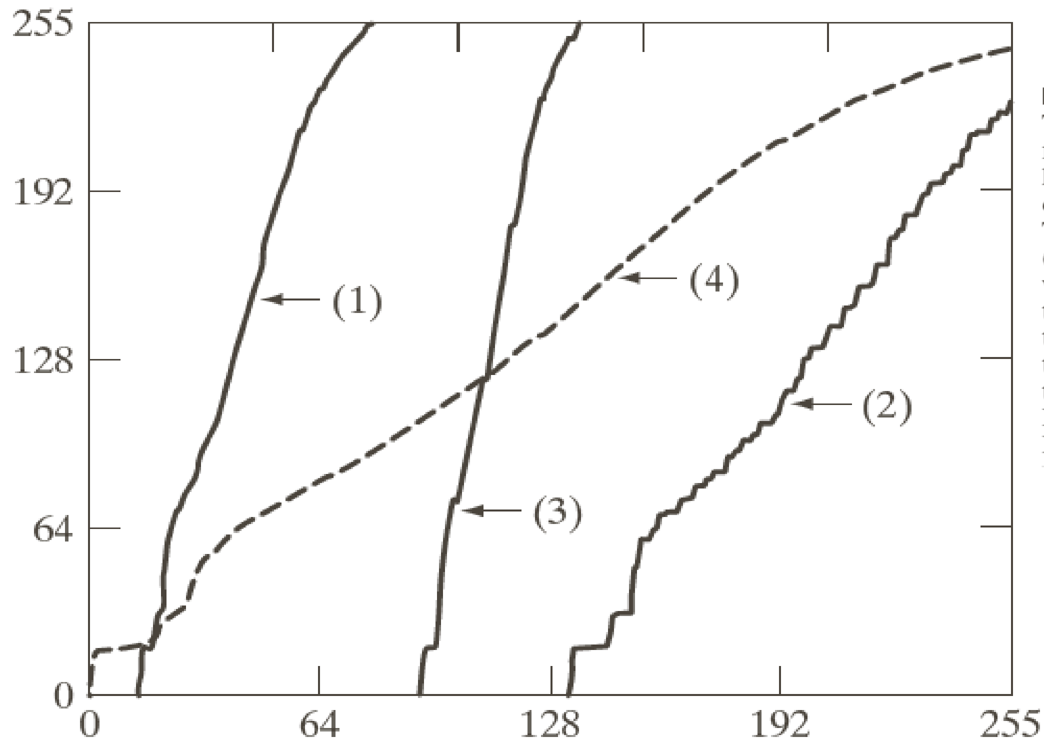
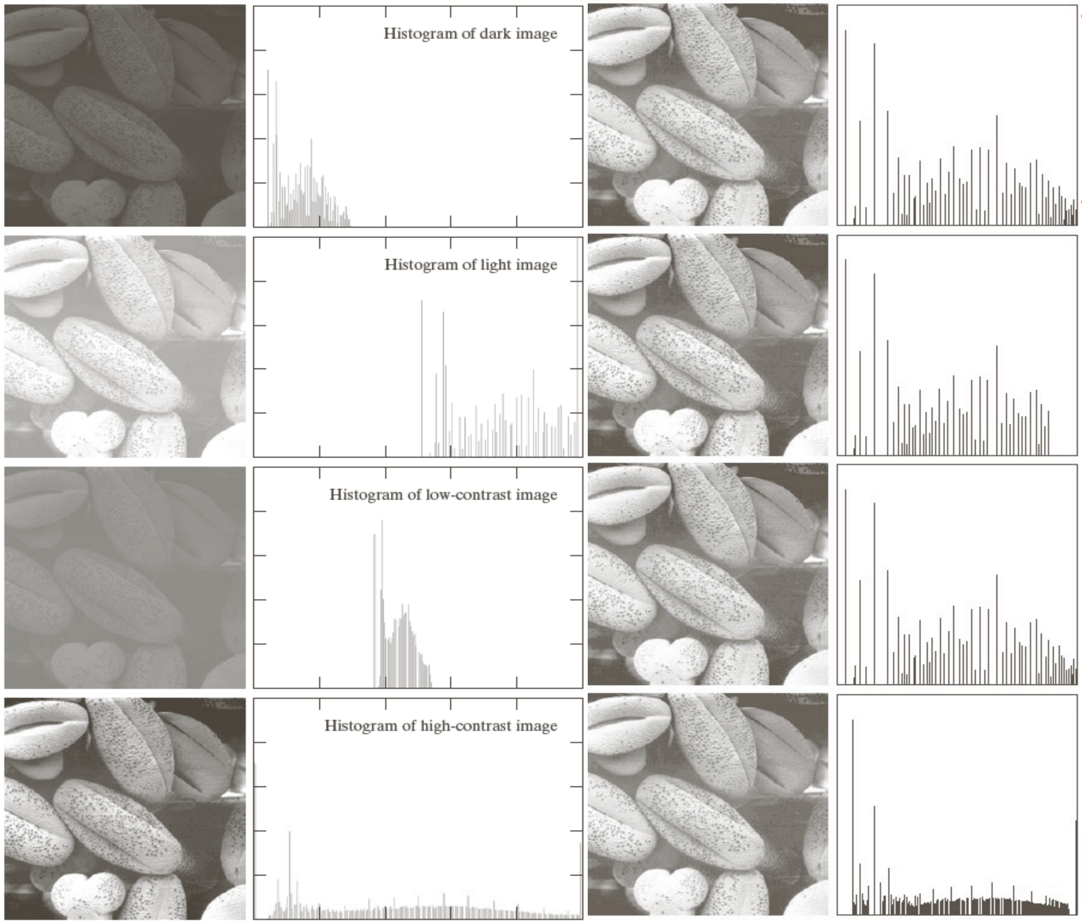


FIGURE 3.21 Transformation functions for histogram equalization. Transformations (1) through (4) were obtained from the histograms of the images (from top to bottom) in the left column of Fig. 3.20 using Eq. (3.3-8).

FIGURE 3.20 Left column: images from Fig. 3.16. Center column: corresponding histogram-equalized images. Right column: histograms of the images in the center column.

Examples

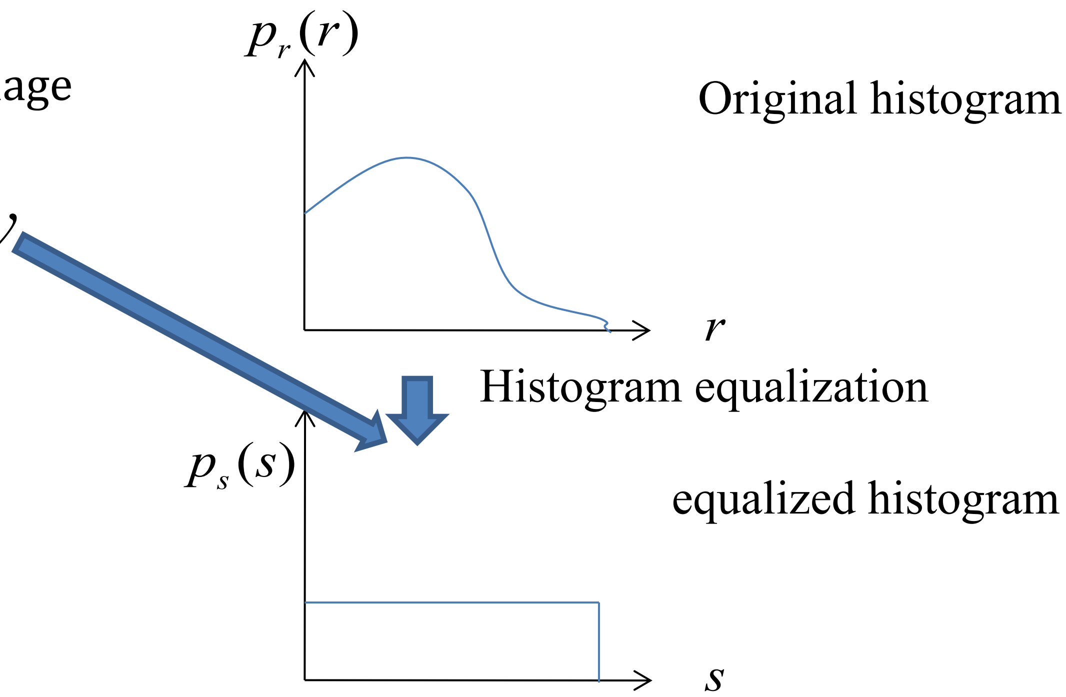


$$s = T(r) = (L-1)\int_0^r p_r(w)dw$$

Histogram Matching (Specification)

Estimated pdf in the input image

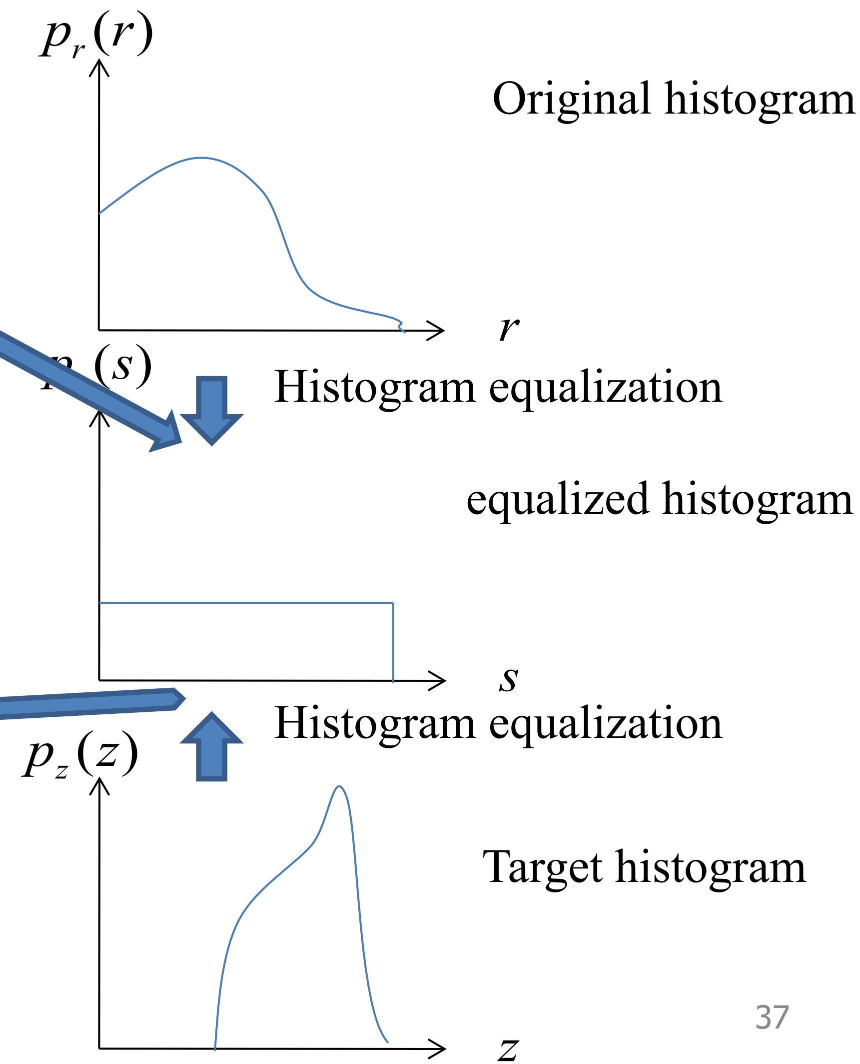
$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$



Histogram Matching (Specification)

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

$$s = G(z) = (L-1)\int_0^z p_z(t)\,dt$$



Histogram Matching (Specification)

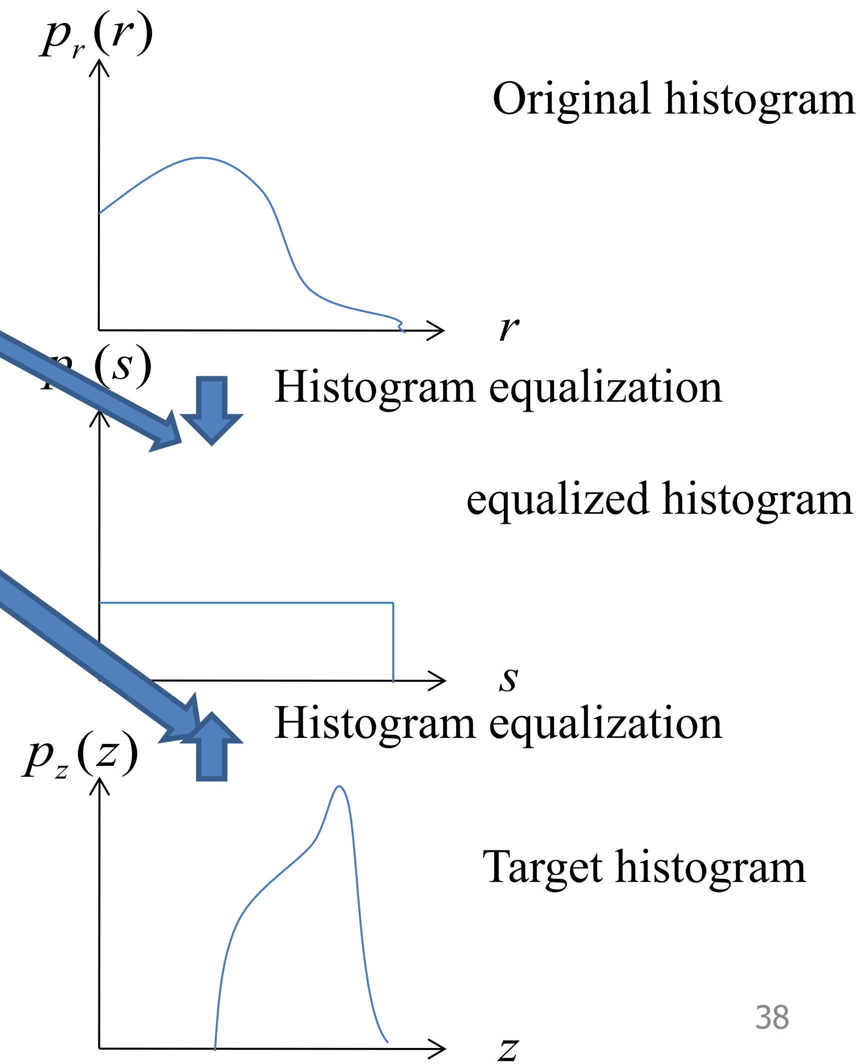
Estimated pdf in the input image

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

$$s = G(z) = (L-1)\int_0^z p_z(t)\,dt$$

Desired pdf in the output image

$$z = G^{-1}(s) = G^{-1}(T(r))$$



Green circles represent known

Red circles represent unknown

Histogram Matching Algorithm for Continuous Data

- Obtain the output image by:
  - First compute the probability distribution function of input data $p_r(r)$
  - Perform histogram equalization → $s=T(r)$
  - Compute $s=G(z)$, where G is the equalization function derived from a specified histogram
  - Perform the inverse mapping
  - The output image with z values is then of the specified histogram

A Continuous Example

$$p_r(r) = \begin{cases} \dfrac{2r}{(L-1)^2} & 0 \le r \le (L-1) \\ 0 & otherwise \end{cases}$$

$$p_z(z) = \begin{cases} \dfrac{3z^2}{(L-1)^3} & 0 \le z \le (L-1) \\ 0 & otherwise \end{cases}$$

Compute z?

A Discrete Example

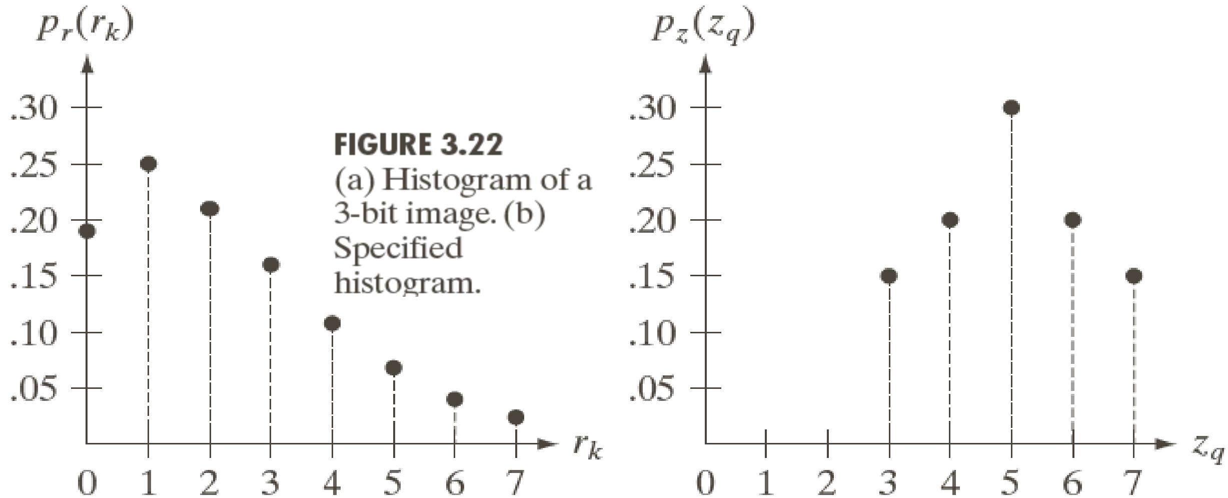


FIGURE 3.22 (a) Histogram of a 3-bit image. (b) Specified histogram.

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

z_q	Specified $p_z(z_q)$
$z_0 = 0$	0.00
$z_1 = 1$	0.00
$z_2 = 2$	0.00
$z_3 = 3$	0.15
$z_4 = 4$	0.20
$z_5 = 5$	0.30
$z_6 = 6$	0.20
$z_7 = 7$	0.15

Histogram Matching Algorithm – Discrete Image

- Discrete histogram require a discretization of the output intensity values

  Step1: Compute histogram of the input image $p_r(r)$ and the histogram equalized image $s = T(r)$

  Step2: Compute $G(z)$ given the desired histogram $p_z(z)$

A Discrete Example – Cont.

r_k	n_k	$p_r(r_k) = n_k/MN$	s
$r_0 = 0$	790	0.19	$S_0=1$
$r_1 = 1$	1023	0.25	$S_1=3$
$r_2 = 2$	850	0.21	$S_2=5$
$r_3 = 3$	656	0.16	$S_3=6$
$r_4 = 4$	329	0.08	$S_4=6$
$r_5 = 5$	245	0.06	$S_5=7$
$r_6 = 6$	122	0.03	$S_6=7$
$r_7 = 7$	81	0.02	$S_7=7$

z_q	Specified $p_z(z_q)$	G(z)
$z_0 = 0$	0.00	$G(z_0)=0$
$z_1 = 1$	0.00	$G(z_1)=0$
$z_2 = 2$	0.00	$G(z_2)=0$
$z_3 = 3$	0.15	$G(z_3)=1$
$z_4 = 4$	0.20	$G(z_4)=2$
$z_5 = 5$	0.30	$G(z_5)=5$
$z_6 = 6$	0.20	$G(z_6)=6$
$z_7 = 7$	0.15	$G(z_7)=7$

Histogram Matching Algorithm – Discrete Image

- Discrete histogram require a discretization of the output intensity values

  Step1: Compute histogram of the input image $p_r(r)$ and the histogram equalized image $s = T(r)$

  Step2: Compute $G(z)$ given the desired histogram $p_z(z)$

  Step3: Given the s_k value, find the value of z_q so that $G(z_q)$ is closest to s_k

A Discrete Example – Cont.

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

s	G(z)	z
$S_0=1$	$G(z_0)=0$	$z_0=0$
$S_1=3$	$G(z_1)=0$	$z_1=1$
$S_2=5$	$G(z_2)=0$	$z_2=2$
$S_3=6$	$G(z_3)=1$	$z_3=3$
$S_4=6$	$G(z_4)=2$	$z_4=4$
$S_5=7$	$G(z_5)=5$	$z_5=5$
$S_6=7$	$G(z_6)=6$	$z_6=6$
$S_7=7$	$G(z_7)=7$	$z_7=7$

Potential issue: Cause a one-to-multiple mapping
-- multiple are mapped to the same
Solution: assign the z-s pair with smallest

Histogram Matching Algorithm – Discrete Image

- Discrete histogram require a discretization of the output intensity values

  Step1: Compute histogram of the input image and the histogram equalized image

  Step2: Compute given the desired histogram

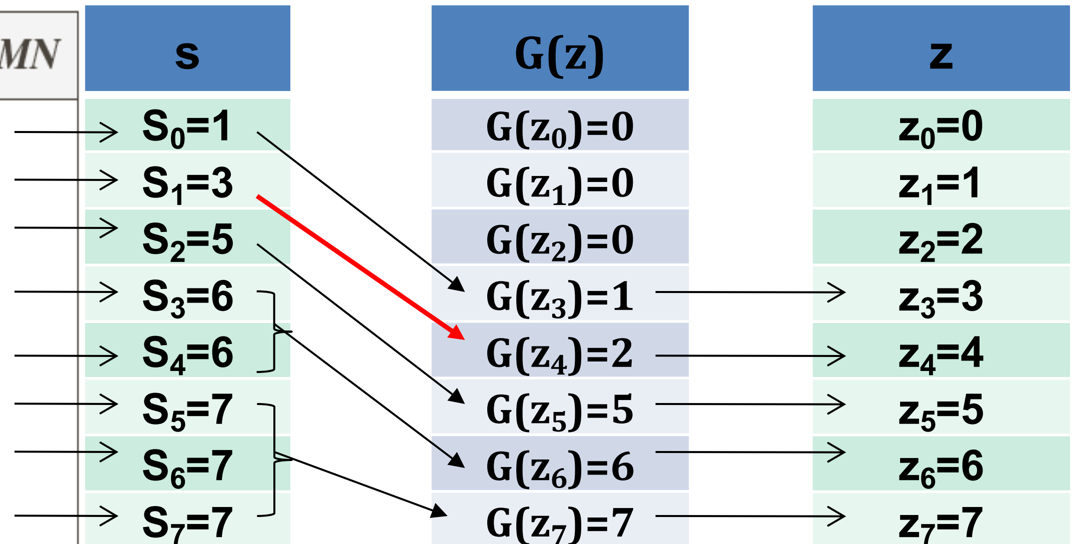
  Step3: Given the value, find the value of so that is closest to

  - Potential issue: Cause a one-to-multiple mapping -- multiple are mapped to the same
  - Solution: assign the z-s pair with smallest

- Step4: form the histogram-specified image using the mapping r-z found above

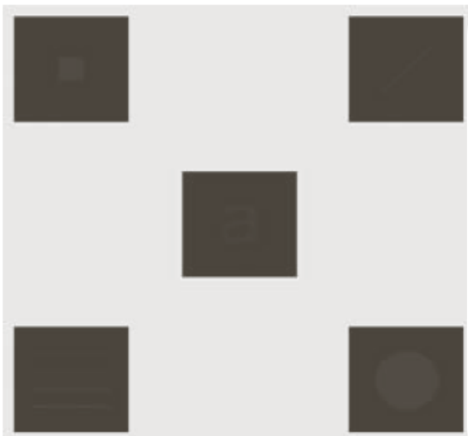
A Discrete Example – Cont.

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

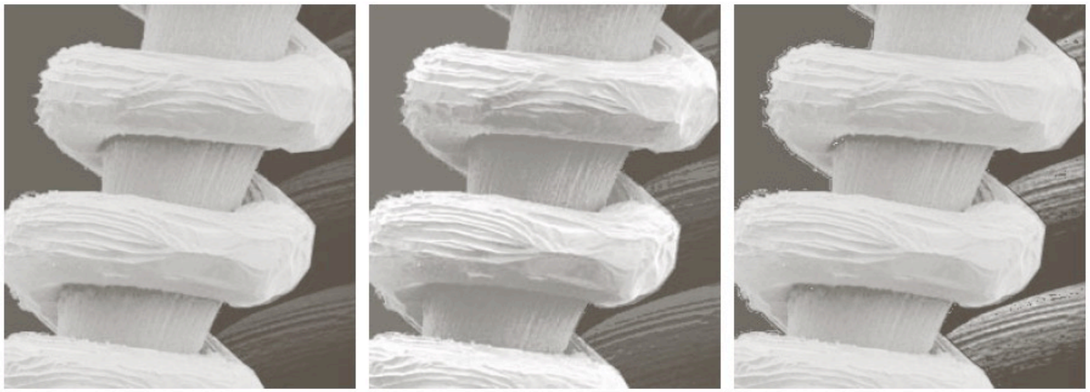


z_q	Specified $p_z(z_q)$	Actual $p_z(z_k)$
$z_0 = 0$	0.00	0.00
$z_1 = 1$	0.00	0.00
$z_2 = 2$	0.00	0.00
$z_3 = 3$	0.15	0.19
$z_4 = 4$	0.20	0.25
$z_5 = 5$	0.30	0.21
$z_6 = 6$	0.20	0.24
$z_7 = 7$	0.15	0.11

Local Histogram Processing



Using Histogram Statistics for Image Enhancement



a b c

FIGURE 3.27 (a) SEM image of a tungsten filament magnified approximately 130×. (b) Result of global histogram equalization. (c) Image enhanced using local histogram statistics. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

$$m_G = \sum_{i=0}^{L-1} r_i p(r_i),$$

$$\sigma_G^2 = \sum_{i=0}^{L-1} (r_i - m)^2 p(r_i)$$

$$m_{S_{xy}} = \sum_{i=0}^{L-1} r_i p_{S_{xy}}(r_i),$$

$$\sigma_{S_{xy}}^2 = \sum_{i=0}^{L-1} (r_i - m_{S_{xy}})^2 p_{S_{xy}}(r_i)$$

$$g(x,y) = \begin{cases} 4f(x,y) & \text{if } m_{S_{xy}} \le 0.4 m_G \text{ AND } 0.02\sigma_G \le \sigma_{S_{xy}} \le 0.4\sigma_G \\ f(x,y) & \text{otherwise} \end{cases}$$

Questions?