UNIVERSITY OF SOUTH CAROLINA

CSCE 574 ROBOTICS

Coordinate Systems

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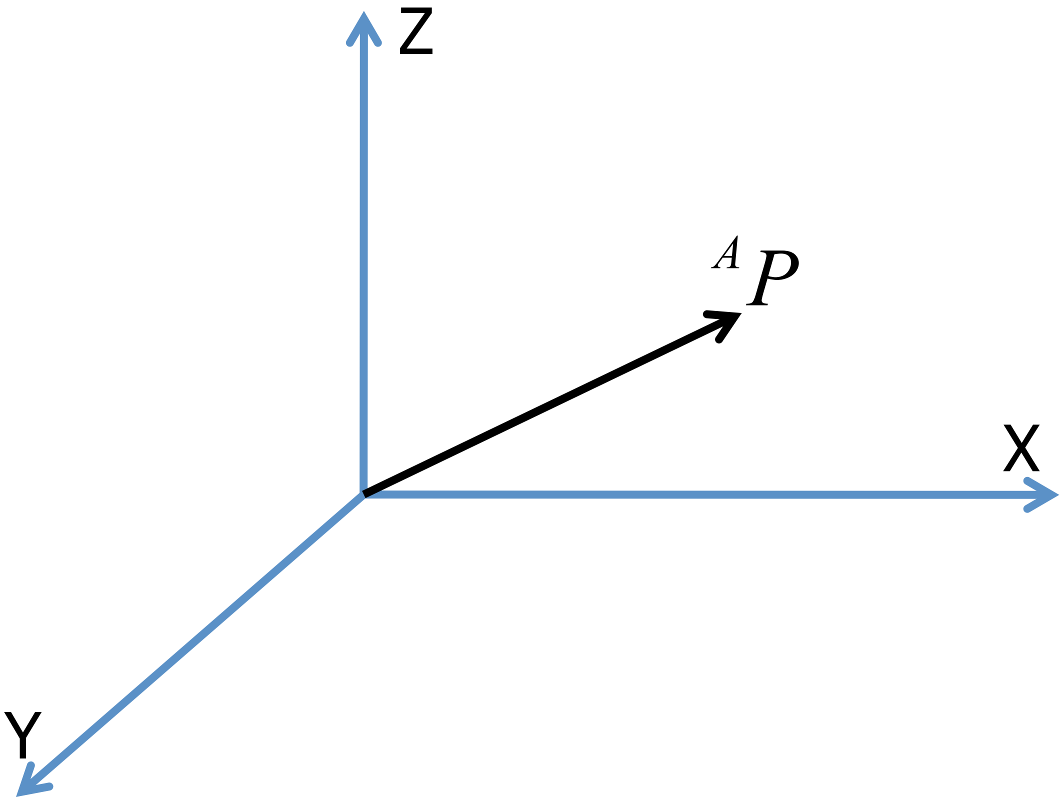
Position Representation

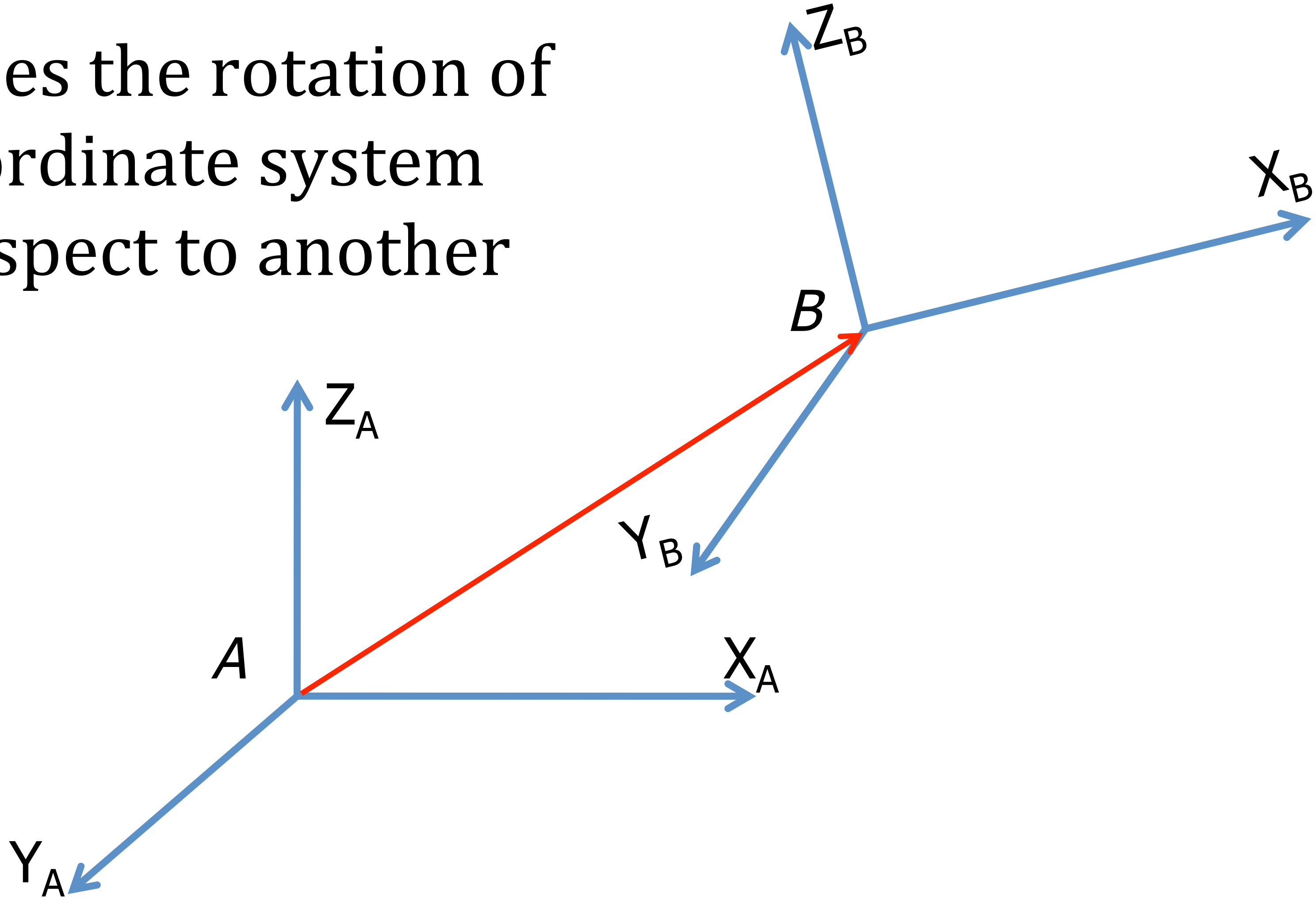
- Position representation is:

$$ {}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} $$



Orientation Representations

- Describes the rotation of one coordinate system with respect to another

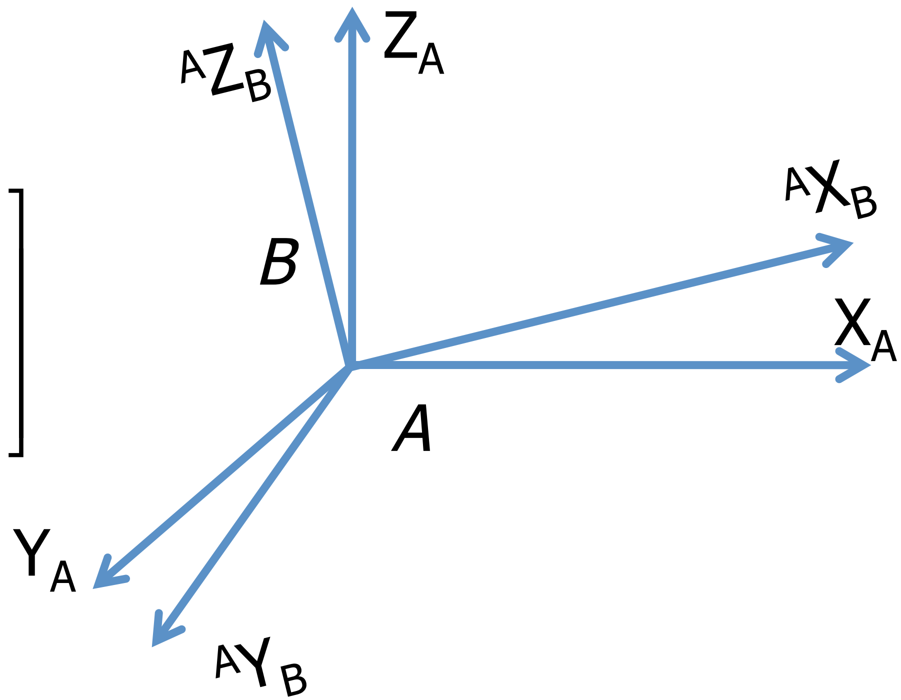


Rotation Matrix

- Write the unit vectors of B in the coordinate system of A.
- Rotation Matrix:

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ = \begin{bmatrix} \hat{X}_{B}\cdot\hat{X}_{A} & \hat{Y}_{B}\cdot\hat{X}_{A} & \hat{Z}_{B}\cdot\hat{X}_{A} \\ \hat{X}_{B}\cdot\hat{Y}_{A} & \hat{Y}_{B}\cdot\hat{Y}_{A} & \hat{Z}_{B}\cdot\hat{Y}_{A} \\ \hat{X}_{B}\cdot\hat{Z}_{A} & \hat{Y}_{B}\cdot\hat{Z}_{A} & \hat{Z}_{B}\cdot\hat{Z}_{A} \end{bmatrix} $$



Properties of Rotation Matrix

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T} $$

$$ {}^{A}_{B}R^{T}\, {}^{A}_{B}R = I_3 $$

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T} $$

Coordinate System Transformation

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0_{3\times1} & 1 \end{bmatrix}$$

where R is the rotation matrix and T is the translation vector

Rotation Matrix

- The rotation matrix consists of 9 variables, but there are many constraints. The minimum number of variables needed to describe a rotation is three.

Rotation Matrix-Single Axis

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Fixed Angles

- One simple method is to perform three rotations about the axis of the original coordinate frame:
  - X-Y-Z fixed angles

$$
{}^A_B R(\theta,\phi,\psi) = R_z(\psi)R_y(\phi)R_x(\theta)
$$

$$
= \begin{bmatrix} \cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi)\sin(\theta)-\sin(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\cos(\theta)+\sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) & \sin(\psi)\sin(\phi)\sin(\theta)+\cos(\psi)\cos(\theta) & \sin(\psi)\sin(\phi)\cos(\theta)+\cos(\psi)\sin(\theta) \\ -\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix}
$$

- There are 12 different combinations

Inverse Problem

- From a Rotation matrix find the fixed angle rotations:

$$
{}^A_B R(\theta,\phi,\psi) = {}^A_B R \Rightarrow
$$

$$
\begin{bmatrix} \cos(\psi)\cos(\phi) & \cos(\psi)\sin(\phi)\sin(\theta) - \sin(\psi)\cos(\theta) & \cos(\psi)\sin(\phi)\cos(\theta) + \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) & \sin(\psi)\sin(\phi)\sin(\theta) + \cos(\psi)\cos(\theta) & \sin(\psi)\sin(\phi)\cos(\theta) + \cos(\psi)\sin(\theta) \\ -\sin(\phi) & \cos(\phi)\sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

thus :

$$
\phi = A\tan 2\left(-r_{31}, \sqrt{\left(r_{11}^2 + r_{21}^2\right)}\right)
$$

$$
\psi = A\tan 2\left({r_{21}}/{\cos(\phi)}, {r_{11}}/{\cos(\phi)} \right)
$$

$$
\theta = A\tan 2\left({r_{32}}/{\cos(\phi)}, {r_{33}}/{\cos(\phi)} \right)
$$

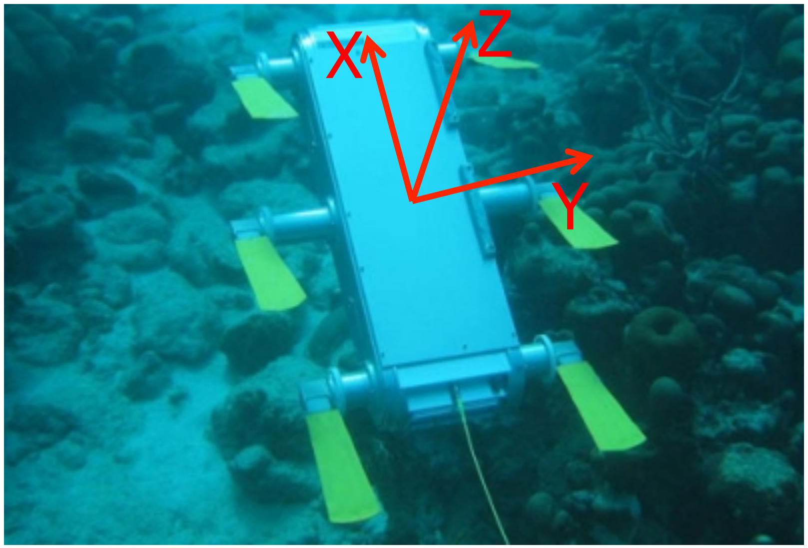
Euler Angles

- **ZYX**: Starting with the two frames aligned, first rotate about the Z_B axis, then by the Y_B axis and then by the X_B axis. The results are the same as with using XYZ fixed angle rotation.

- There are 12 different combination of Euler Angle representations

Euler Angles

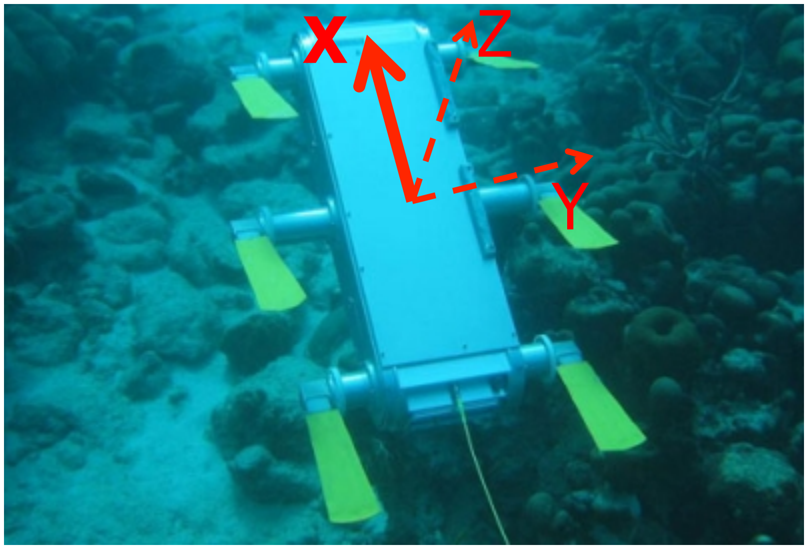
- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw



Euler Angles

- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw

Roll



Euler Angles

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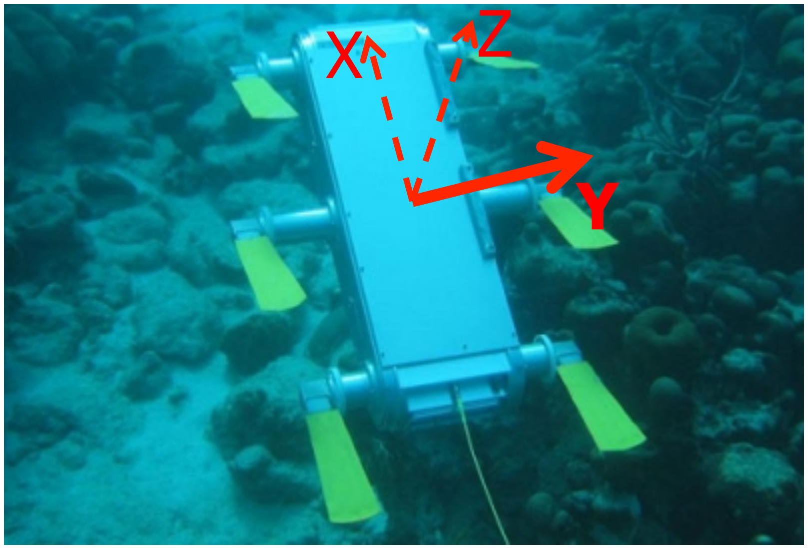
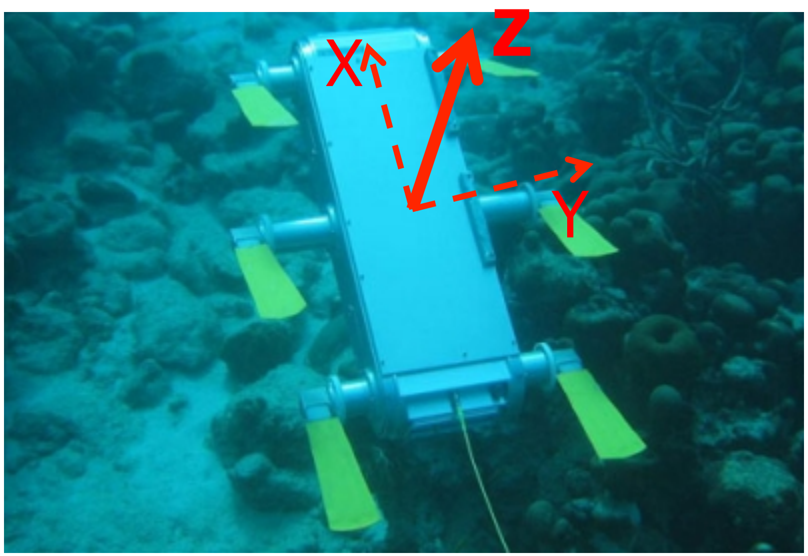
Pitch



Euler Angles

- Traditionally the three angles along the axis are called Roll, Pitch, and Yaw

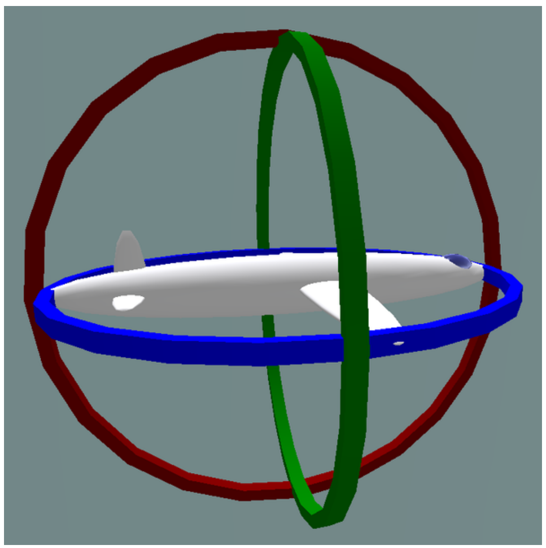
Yaw

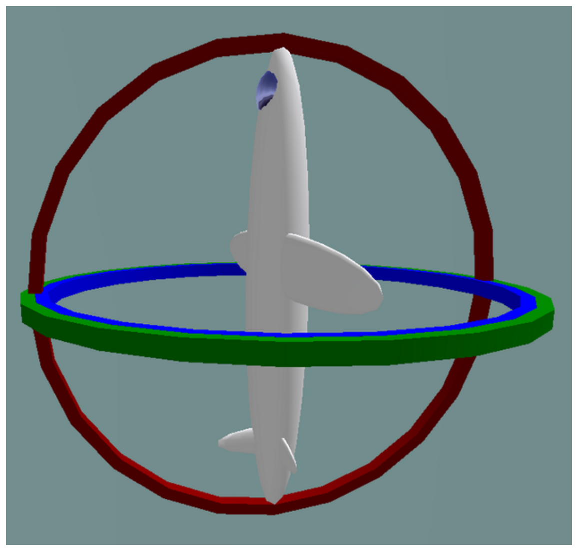


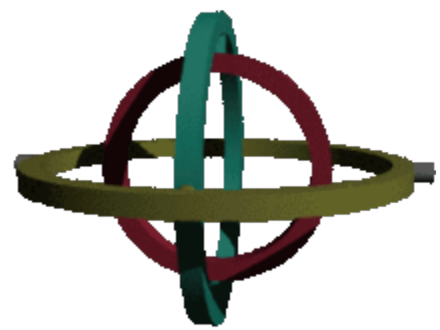
Euler Angle concerns: Gimbal Lock

Using the **ZYZ** convention

- $(90°, 45°, -105°) \equiv (-270°, -315°, 255°)$ multiples of 360°
- $(72°, 0°, 0°) \equiv (40°, 0°, 32°)$ singular alignment (Gimbal lock)
- $(45°, 60°, -30°) \equiv (-135°, -60°, 150°)$ bistable flip

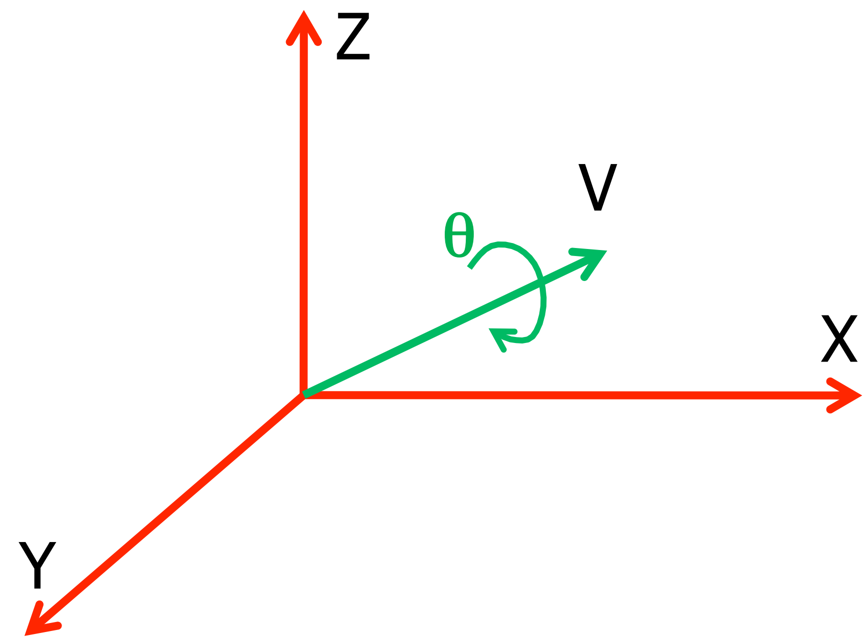






Axis-Angle Representation

- Represent an arbitrary rotation as a combination of a vector and an angle



Quaternions

- Are similar to axis-angle representation
- Two formulations
  - Classical
  - Based on JPL's standards

    W. G. Breckenridge, "Quaternions - Proposed Standard Conventions," JPL, Tech. Rep. INTEROFFICE MEMORANDUM IOM 343-79-1199, 1999.
- Avoids Gimbal lock

- See also: M. D. Shuster, "A survey of attitude representations," Journal of the Astronautical Sciences, vol. 41, no. 4, pp. 439–517, Oct.–Dec. 1993.

Quaternions

	Classic notation	JPL-based
	$\bar{q} = q_4 + q_1 i + q_2 j + q_3 k$	$\bar{q} = q_4 + q_1 i + q_2 j + q_3 k$
	$i^2 = j^2 = k^2 = ijk = -1$ $ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$	$i^2 = j^2 = k^2 = -1$ $-ij = ji = k, -jk = kj = i, -ki = ik = j$
Vector Notation	$\bar{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix},\ q_0 = \cos\left(\theta/2\right),\ \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \sin\left(\theta/2\right)\cos(\beta_x) \\ \sin\left(\theta/2\right)\cos(\beta_y) \\ \sin\left(\theta/2\right)\cos(\beta_z) \end{bmatrix}$	$\bar{q} = \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix},\ \mathbf{q} = \begin{bmatrix} k_x \sin\left(\theta/2\right) \\ k_y \sin\left(\theta/2\right) \\ k_z \sin\left(\theta/2\right) \end{bmatrix},\ q_4 = \cos\left(\theta/2\right)$
		$\lVert\bar{q}\rVert = 1,\ \bar{q} \otimes \bar{p},\ \mathbf{q} \times \mathbf{p},\ \bar{q}_I,\ \lfloor \mathbf{q} \times \rfloor$

See also: N. Trawny and S. I. Roumeliotis, “Indirect Kalman Filter for 3D Attitude Estimation,” University of Minnesota, Dept. of Comp. Sci. & Eng., Tech. Rep. 2005-002, March 2005.

Coordinate frames on PR2