Today’s Agenda

Geometry

Geometry

Three basic elements

- Points
- Scalars
- Vectors

Three type of spaces

- (Linear) vector space: scalars and vectors
- Affine space: vector space + points
- Euclidean space: vector space + distance

Euclidean Spaces

Distance (scalar) + a vector space

Operations

- Vector-vector addition
- Scalar-vector multiplication
- Scalar-scalar operations
- Inner (dot) products

$v \cdot v > 0 \text{ if } v \neq 0$ → Magnitude (length) of a vector

$$|v| = \sqrt{v \cdot v}$$

Distance between two points P & Q

$$|P - Q| = \sqrt{(P - Q) \cdot (P - Q)}$$

Angle between two vectors v & u → $\cos\theta = \frac{u \cdot v}{|u||v|}$

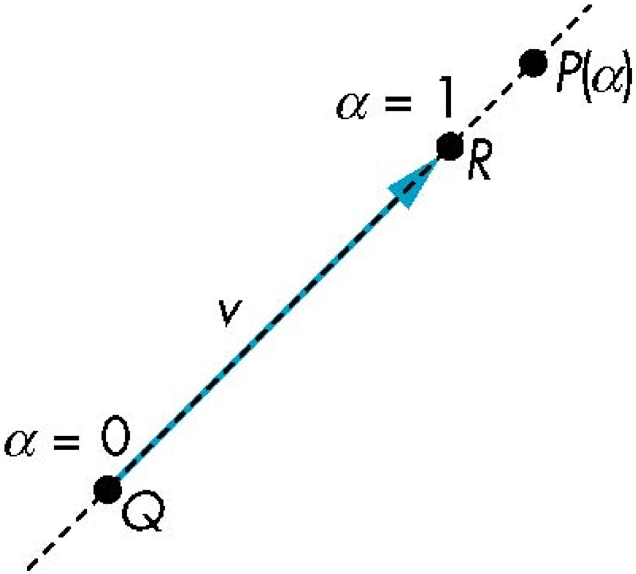
Line Segments

If we use two points to define v, then

$$P(\boldsymbol{\alpha}) = Q + \boldsymbol{\alpha} v = Q + \boldsymbol{\alpha}(R - Q) = \boldsymbol{\alpha} R + (1 - \boldsymbol{\alpha})Q$$

For $0 \leq \boldsymbol{\alpha} \leq 1$ we get all the points on the *line segment*

joining R and Q



Affine Sums and Convex Hull

Consider the "sum"

$$P = \boldsymbol{\alpha}_1 P_1 + \boldsymbol{\alpha}_2 P_2 + \cdots + \boldsymbol{\alpha}_n P_n$$

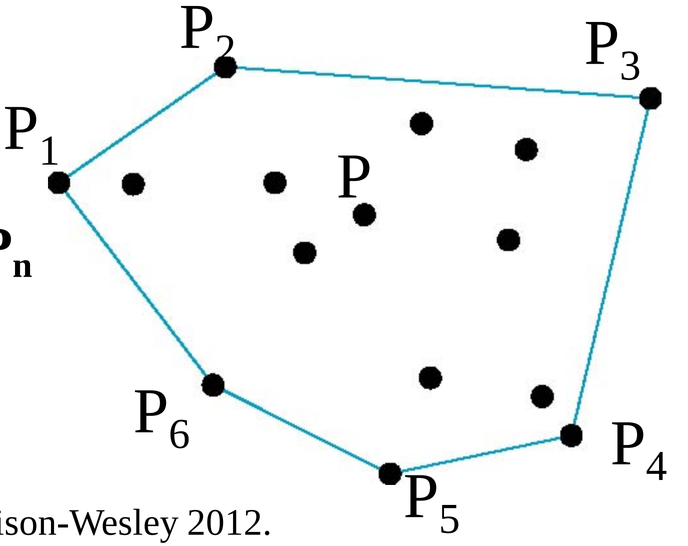
Can show by induction that this sum makes sense iff

$$\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 + \cdots + \boldsymbol{\alpha}_n = 1$$

in which case we have the *affine sum* of the points $P_1, P_2, \ldots P_n$

If, in addition, $\alpha_i >= 0$, we have the *convex hull* of $P_1, P_2, \ldots P_n$

Smallest convex object containing $P_1, P_2, \ldots P_n$



Planes

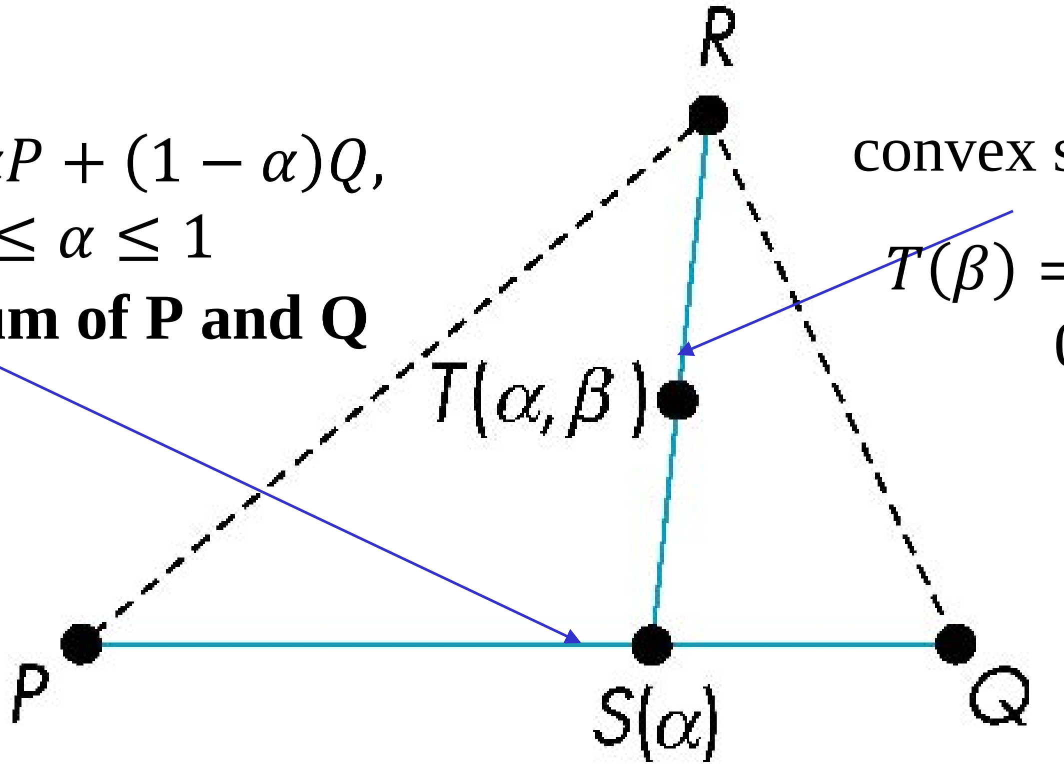
A plane can be defined by three non-collinear points

$$S(\alpha) = \alpha P + (1-\alpha)Q,$$
$$0 \le \alpha \le 1$$

convex sum of P and Q

convex sum of S(α) and R

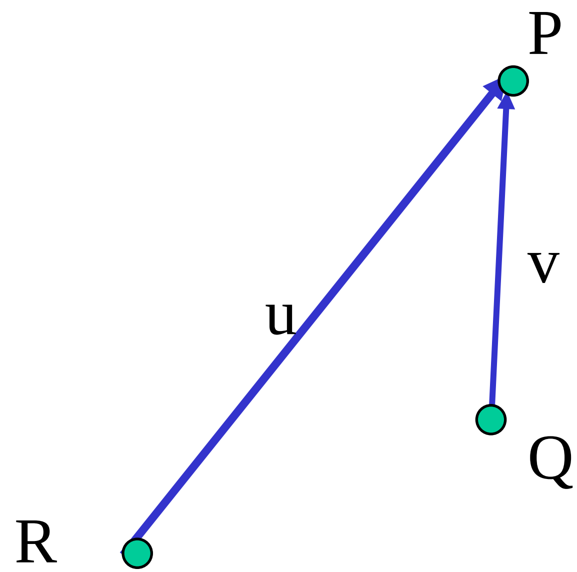
$$T(\beta) = \beta S + (1-\beta)\mathrm{R}$$
$$0 \le \beta \le 1$$



$$T(\alpha,\beta) = \beta[\alpha P + (1-\alpha)Q] + (1-\beta)\mathrm{R}$$
$$0 \le \alpha, \beta \le 1$$

Planes

A plane can also be defined by a point and two vectors



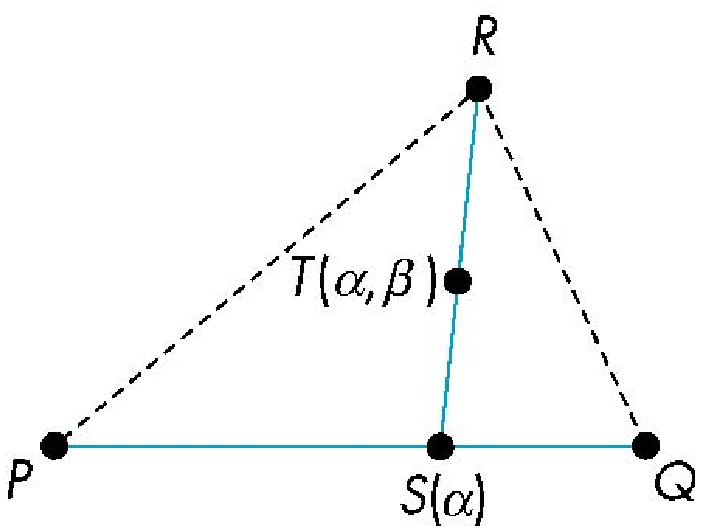
$$T(\alpha, \beta) = P + \beta(1 - \alpha)(Q - P) + (1 - \beta)(R - P)$$

$$\Rightarrow \quad T(\alpha', \beta') = P + \alpha' u + \beta' v$$

Parametric form of planes

Triangles

A triangle can be defined by three nonclinear points



Triangle is convex so any point inside can be represented as an affine sum

Affine sum

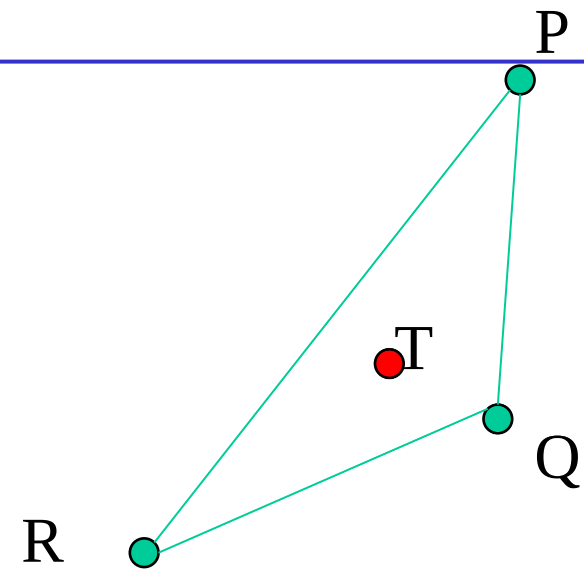
$$T(\alpha,\beta) = \alpha\beta P + \beta(1-\alpha)Q + (1-\beta)R$$

$$\Rightarrow \quad T(\alpha',\beta',\gamma') = \alpha' P + \beta' Q + \gamma' R$$

where

$$0 \le \alpha',\beta',\gamma' \le 1$$
$$\alpha' + \beta' + \gamma' = 1$$

Barycentric Coordinates



$T(a_1, a_2, a_3) = a_1 P + a_2 Q + a_3 R$

where

$a_1 + a_2 + a_3 = 1$

$a_i >= 0$

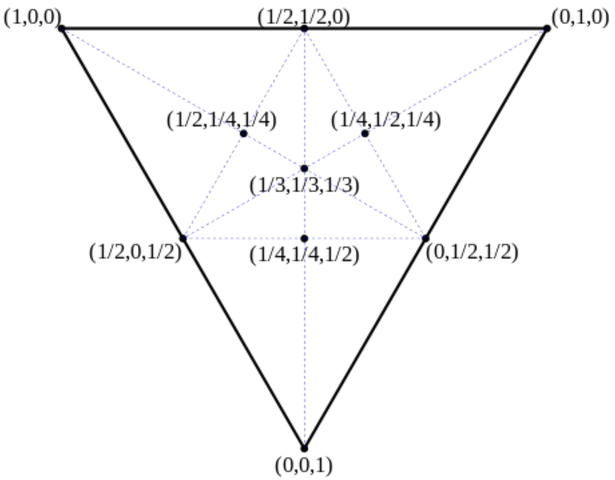
The representation is called the barycentric coordinate representation of P

Where is T if $a_i<0$? T is outside of the triangle, but on the same plane formed by P, Q, and R

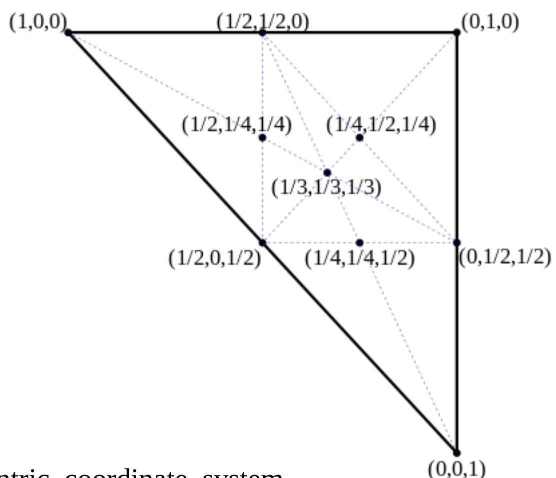
Where is T if $a_1 + a_2 + a_3 \neq 1$? T is not on the same plane PQR

Barycentric Coordinates

Equilateral triangle



right triangle



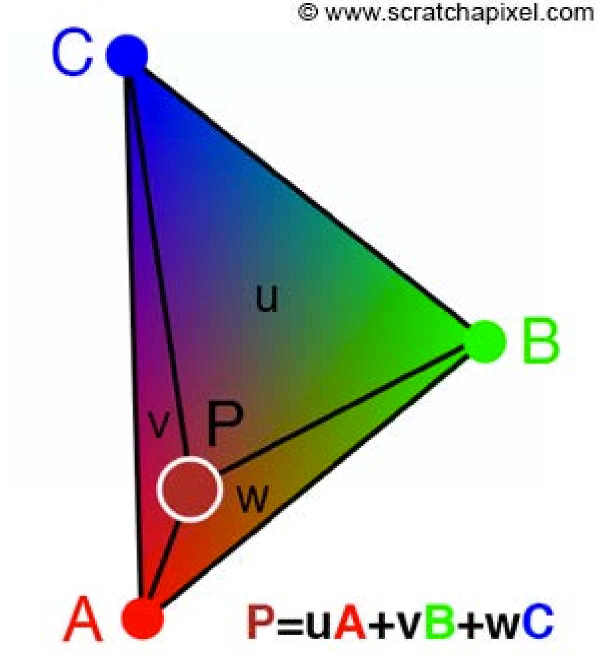
https://en.wikipedia.org/wiki/Barycentric_coordinate_system

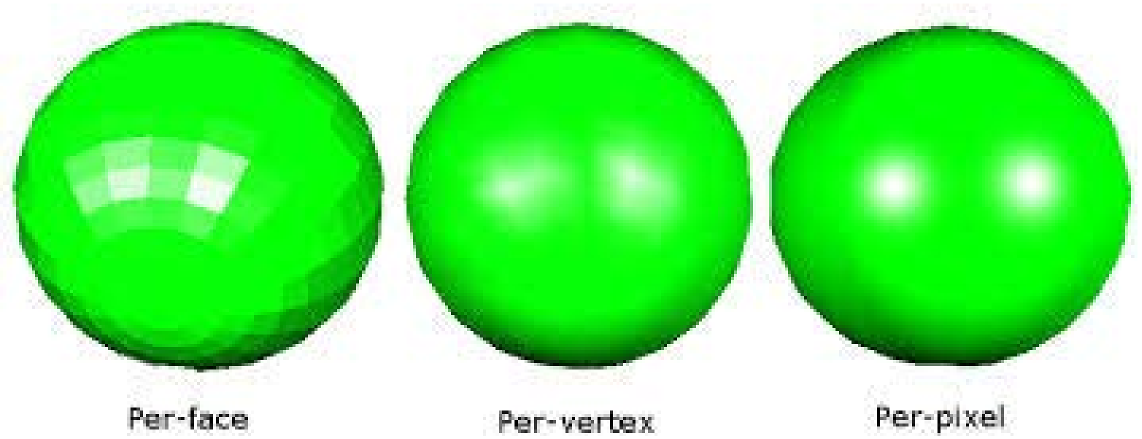
Barycentric Coordinates

Ray-triangle intersection:

testing if a ray intersects with a triangle

Rendering a triangle





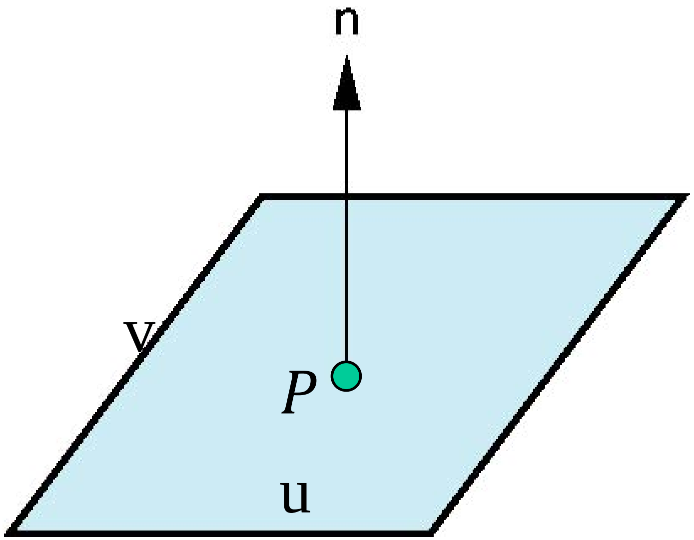
courses.cs.washington.edu

Normals

Every plane has a vector n normal (perpendicular) to it

From point-two vector form $T(\alpha, \beta) = P + \alpha u + \beta v$, we know we can use the cross product to find $n = u \times v$ and the equivalent form

$$(T - P) \cdot n = 0$$



3D Primitives

So far, we have discussed how to represent 2D geometric objects, the primitives

How about primitives in 3D?

For existing graphics hardware and software, three features describe the 3D objects

- 2D surfaces
- vertices
- Flat, convex polygons, especially triangles

Reference Frame and Coordinate Systems

Until now we have been able to work with geometric entities without using any frame of reference, i.e., in a coordinate-free system

- **Coordinate system**
- **Reference frame**

Coordinate Systems

Consider a basis $v_1, v_2, \ldots, v_n$ of $\mathcal{R}^n$

A vector in $\mathcal{R}^n$ is written $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

The list of scalars $\{\alpha_1, \alpha_2, \ldots. \alpha_n\}$ is the *representation* of v with respect to the given basis

We can write the representation as a row or column array of scalars

$$\boldsymbol{a} = [\alpha_1 \alpha_2 \cdots \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_n \end{bmatrix}$$

Example

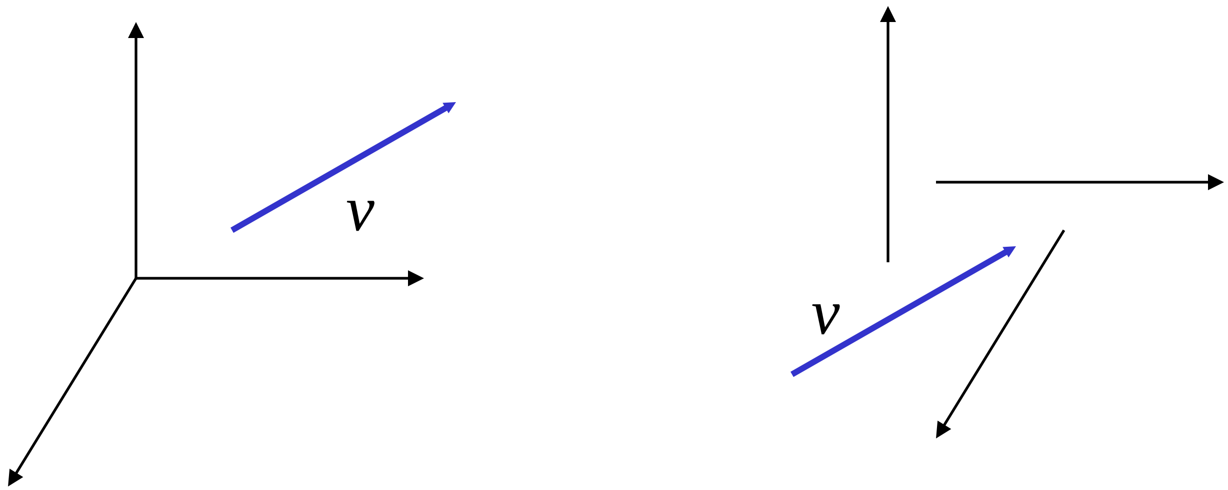
$\mathbf{v}=2\mathbf{v}_1+3\mathbf{v}_2-4\mathbf{v}_3$

$$\boldsymbol{a} = [2\ 3\ -4]^T$$

Note that this representation is with respect to a particular basis

Coordinate Systems

Which is correct?



Both,

because coordinate system is defined in vector space and vectors have no fixed location

Frame of Reference

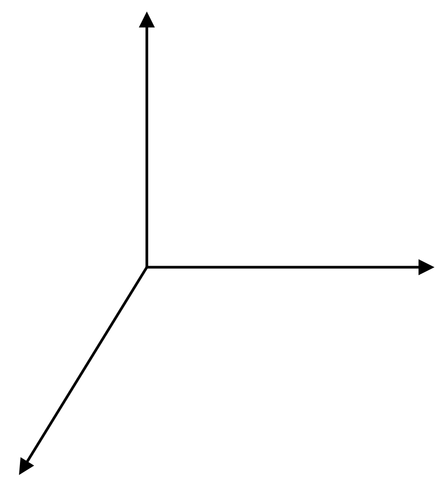
Need a frame of reference to relate points and objects to our physical world. For example, where is a point? Can't answer without a reference system

A coordinate system is insufficient to represent points

Adding a reference point (origin) to a coordinate system

→ **Frame defined in affine space**

- Frames used in graphics
  - World frame
  - Camera frame
  - Image frame



Representation in a Frame

Frame determined by (P_0, v_1, v_2, v_3)

Within this frame, every vector can be written as

$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ → e.g., a 3-tuples

Every point can be written as

$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$

$$\mathbf{e_1} = (1, 0, 0)^T,$$

$$\mathbf{e_2} = (0, 1, 0)^T,$$

$$\mathbf{e_3} = (0, 0, 1)^T.$$

Confusing Points and Vectors

Consider the point and the vector

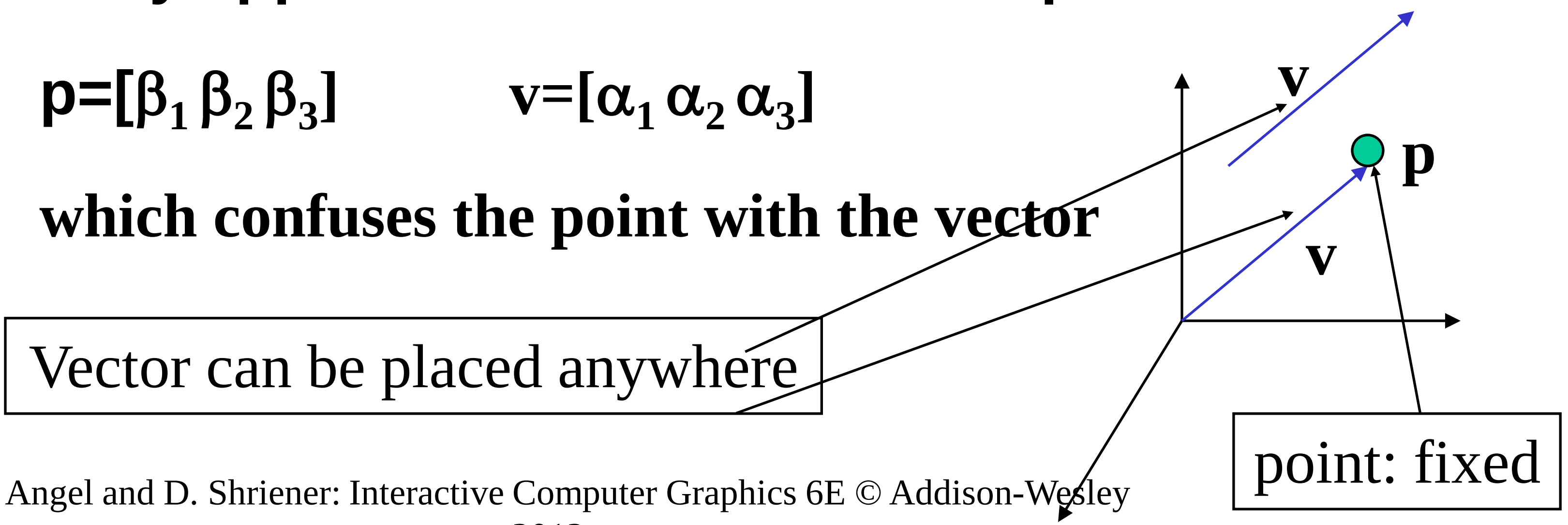
$$\mathbf{P} = \mathbf{P}_0 + \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

They appear to have the similar representations

$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3]$ $\qquad$ $\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3]$

which confuses the point with the vector



A Single Representation - Homogeneous Coordinates

If we define $0 \bullet P = 0$ and $1 \bullet P = P$ then we can write

$$\mathbf{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [\alpha_1\ \alpha_2\ \alpha_3\ 0\ ]\ [v_1\ v_2\ v_3\ P_0]^T$$

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = [\beta_1\ \beta_2\ \beta_3\ 1\ ]\ [v_1\ v_2\ v_3\ P_0]^T$$

Thus we obtain the four-dimensional *homogeneous coordinate* representation

$$\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3\ 0\ ]^T$$

$$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3\ 1\ ]^T$$

Homogeneous Coordinates

In general, the homogeneous coordinates for a 3D point [x y z] is given as

$$P = [x' \; y' \; z' \; w]^T = [wx \; wy \; wz \; w]^T$$

When w≠0, we return to a 3D point by $\mathrm{P} = [x \; y \; z \; 1]$

where $x \leftarrow x'/w, y \leftarrow y'/w, z \leftarrow z'/w$

If w=0, the representation is that of a vector

Homogeneous Coordinates and Computer Graphics

Homogeneous coordinates are key to all computer graphics systems

- All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 matrices
- Hardware pipeline works with 4 dimensional representations
- For orthographic viewing, we can maintain $w=0$ for vectors and $w=1$ for points
- For perspective, we need a *perspective division*

Change of Coordinate Systems

Consider two representations of the same vector $\boldsymbol{d}$ with respect to two different bases. The representations are

$$\mathbf{a}=[\alpha_1\ \alpha_2\ \ \alpha_3\ ]$$
$$\mathbf{b}=[\beta_1\ \ \beta_2\ \ \beta_3]$$

where

$$d=\alpha_1 v_1+\ \alpha_2 v_2\ +\alpha_3 v_3 = [v_1\ v_2\ v_3]\ [\alpha_1\ \alpha_2\ \alpha_3]^T$$
$$=\beta_1 u_1+\ \beta_2 u_2\ +\beta_3 u_3 = [u_1\ u_2\ u_3]\ [\beta_1\ \beta_2\ \beta_3]^T$$

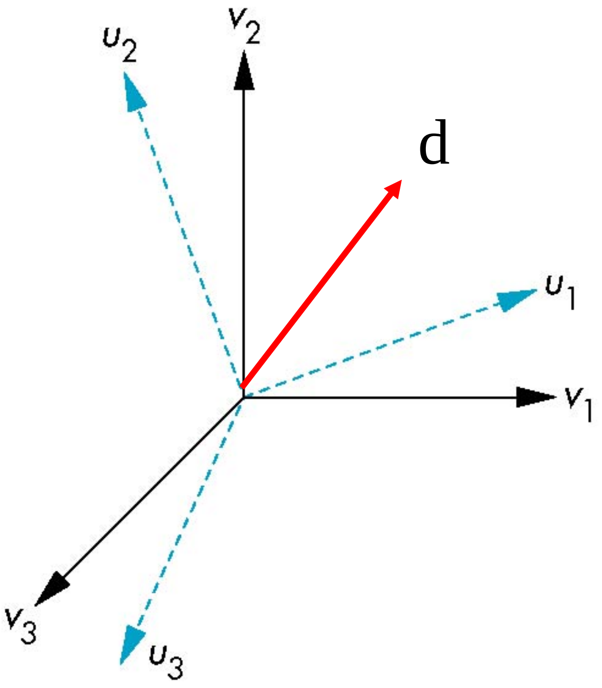
Let $\mathbf{V}=[v_1\ v_2\ v_3]$ and $\mathbf{U}=[u_1\ u_2\ u_3]$

$$\boldsymbol{d} = \mathbf{V}\boldsymbol{a} = \mathbf{U}\boldsymbol{b}$$

Representing second basis in terms of first

Each of the basis vectors, u1,u2, u3, are vectors that can be represented in terms of the first basis

$u_1 = \gamma_{11}v_1+\gamma_{12}v_2+\gamma_{13}v_3$
$u_2 = \gamma_{21}v_1+\gamma_{22}v_2+\gamma_{23}v_3$
$u_3 = \gamma_{31}v_1+\gamma_{32}v_2+\gamma_{33}v_3$



Changing Representations

The coefficients define a 3 x 3 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

and the bases can be related by

$$\mathbf{U}=\mathbf{V}\mathbf{M}^{T}$$

We can change the representation from one coordinate system to the other as

$$\mathbf{a}=\mathbf{M}^{T}\mathbf{b} \quad \text{and} \quad \mathbf{b}=(\mathbf{M}^{T})^{-1}\mathbf{a}$$

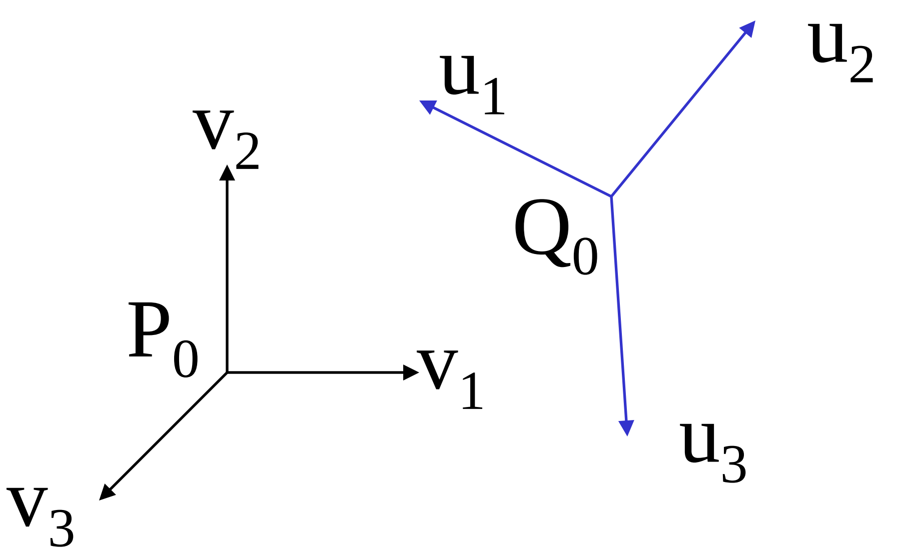
Change of Frames

We can apply a similar process in homogeneous coordinates to the representations of both points and vectors

Consider two frames:

(P_0, v_1, v_2, v_3)

(Q_0, u_1, u_2, u_3)



Any point or vector can be represented in either frame

We can represent Q_0, u_1, u_2, u_3 in terms of P_0, v_1, v_2, v_3

Representing One Frame in Terms of the Other

Extending what we did with change of bases

$$\mathbf{u}_1 = \gamma_{11}\mathbf{v}_1 + \gamma_{12}\mathbf{v}_2 + \gamma_{13}\mathbf{v}_3$$
$$\mathbf{u}_2 = \gamma_{21}\mathbf{v}_1 + \gamma_{22}\mathbf{v}_2 + \gamma_{23}\mathbf{v}_3$$
$$\mathbf{u}_3 = \gamma_{31}\mathbf{v}_1 + \gamma_{32}\mathbf{v}_2 + \gamma_{33}\mathbf{v}_3$$
$$\mathbf{Q}_0 = \gamma_{41}\mathbf{v}_1 + \gamma_{42}\mathbf{v}_2 + \gamma_{43}\mathbf{v}_3 + \mathbf{P}_0$$

defining a 4 x 4 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix} \Rightarrow [\mathbf{U} \quad Q_0] = [\mathbf{V} \quad P_0]\mathbf{M}^T$$

Changing Representations

Any point or vector has a representation in a frame

$\mathbf{a}=[\alpha_1\ \alpha_2\ \ \alpha_3\ \alpha_4\,]$ in the first frame
$\mathbf{b}=[\beta_1\ \beta_2\ \ \beta_3\ \beta_4\,]$ in the second frame

where $\alpha_4 = \beta_4 = 1$ for points and $\alpha_4 = \beta_4 = 0$ for vectors

We can change the representation from one frame to the other as

$$\mathbf{a}=\mathbf{M}^T\mathbf{b} \quad \text{and} \quad \mathbf{b}=(\mathbf{M}^T)^{-1}\mathbf{a}$$

The matrix $\mathbf{M}$ is 4 x 4 and specifies an affine transformation in homogeneous coordinates