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Note Title 2011-03-15

Predicate Logic (or: First-order Logic [FOL]
or First-order predicate logic
or Predicate Calculus)

P_i^k k is the arity of a predicate (symbol)
arity is the number of arguments

Terms

Ex. 43 matrix: $(Q(x) \vee (P(f(x), z) \wedge Q(a)) \vee R(x, z, g(x)))$

Shortcut:

$I_a(x) = x^a$

$I_a(f) = f^a$

$I_a(P) = P^a$

$I_a(b) = b^a$

$P(x, f(x))$

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$I_a(P(x, f(x))) =$

$= P^a(x^a, f^a(x^a)) =$

$= <(2, succ(2)) = <(2, 3)$

$= 2 < 3 = 0 \quad (f)$

If F has the form $F = \forall x G$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if for all } u \in U_{\mathcal{A}},\ \mathcal{A}_{[x/u]}(G) = 1 \\ 0, & \text{otherwise} \end{cases}$$

If F has the form $F = \exists x G$, then

$$\mathcal{A}(F) = \begin{cases} 1, & \text{if for some } u \in U_{\mathcal{A}},\ \mathcal{A}_{[x/u]}(G) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$\mathcal{A} \models F$ iff $\mathcal{A}(F) = 1$

If $\mathcal{A}(F) = 1$ for every suitable structure $\mathcal{A}$, then F is valid, written $\models F$

If there is some suitable structure $\mathcal{A}$ for which $\mathcal{A}(F) = 1$, then F is satisfiable.

If there is no model for F (i.e., no suitable structure $\mathcal{A}$ s.t. $\mathcal{A}(F) = 1$), then F is unsatisfiable (or a contradiction).

Exercise 44

$F: \forall x \exists y \, P(x, y, f(z))$

A model for F is $\mathcal{A}(U_\mathcal{A}, I_\mathcal{A})$, where

$U_\mathcal{A} = \{c\}$ $\quad z^\mathcal{A} = c$

$f^\mathcal{A} = c \to c$ $\quad P^\mathcal{A} = \{(c, c, c)\}$

An interpretation $\mathcal{B}$ s.t. $\mathcal{B}(F) = 0$

$U_{\mathcal{B}} = \{c\}$ $\qquad z^{\mathcal{B}} = c$

$f^{\mathcal{B}} = c \to c$ $\qquad p^{\mathcal{B}} = \{\ \}$

Exercise 45

$\mathfrak{a}$ is a model of F_1, F_2, and F_3

$U = \{1, 2, 3\}$

$F_1 + F_2 + F_3$ $\quad p^{\mathfrak{a}} = \{(1,1), (2,2), (3,3),$
$(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

F_2 + F_3 $\mathcal{B}$ is a model of F_2, F_3, not F_1

$p^{\mathcal{B}} = \{\}$

F_1, F_3 $\mathcal{C}$ is a model of F_1, F_3, not F_2

$p^{\mathcal{C}} = \{(1,1), (2,2), (3,3), (1,2)\}$

F_1, F_2 $\mathcal{D}$ is a model of F_1, F_2 not F_3

$\{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$

Exercise 46

Syntax is changed by adding this:

if t_1 and t_2 are terms, then $t_1 = t_2$ is a formula

Semantics are changed by adding:

if F has the form $t_1 = t_2$, then

$$a(F) = \begin{cases} 1 & \text{if } a(t_1) = a(t_2) \\ 0 & \text{otherwise} \end{cases}$$

Exercise 47

For part (a), the question is whether the system of inequalities

$$\begin{cases} x < y \\ z < y \\ x < z \\ z \geq x \end{cases}$$

(yes)

$x = 1$
$z = 2$
$y = 3$

(b)
$$\begin{cases} y = x+1 \\ y = z+1 \\ z = x+1 \\ x \neq z+1 \end{cases}$$

(no)

Exercise 49

$F = \forall x\, E(x,x) \wedge \exists x \exists y \exists z$
$(\neg E(x,y) \wedge \neg E(x,z) \wedge \neg E(y,z))$

Here is instead a formula whose models have universes of cardinality exactly 3: (Note: $t_1 \neq t_2$ is an abbreviation for $\neg(t_1 = t_2)$)

$$\exists x \exists y \exists z \forall u \underbrace{(x \neq y \wedge y \neq z \wedge x \neq z)}_{\text{at least 3 individuals}} \wedge \underbrace{(x = u \vee y = u \vee z = u)}_{\text{at most three individuals}}$$

(individual means element of the universe)

Exercise 51

$$F = \forall x \forall y \forall z \left((x=y) \lor (y=z) \lor (x=z) \right)$$

Exercise 52

(a) $\forall x \forall y \, \neg \left(P(x,y) \land P(y,x) \right)$

(b) $\forall x \forall y \left((f(x) = f(y)) \to (x = y) \right)$ f is an injective fcn.
(one-to-one)

(c) $\forall y \exists x \, (f(x) = y)$ f is onto or surjective

A function both injective and surjective
(one-to-one) (onto)

is called a bijection

one-to-one correspondence

Exercise 53

$F = \forall x \forall y \forall z \left(f(x, f(y,z)) = f(f(x,y), z)\right)$ (Associativity)

$\wedge \exists x \left[\forall y \left(f(x,y) = y\right) \right.$ (neutral element)

$\left. \wedge \forall y \exists z \left(f(y,z) = x \right) \right]$ (inverse)

Exercise 54.

$$
\begin{aligned}
F = {} & \mathrm{IsEmpty}(\mathrm{nullstack}) \\
& \wedge \forall x \forall y \; \neg \mathrm{IsEmpty}(\mathrm{push}(x,y)) \\
& \wedge \forall x \forall y \; (\mathrm{top}(\mathrm{push}(x,y) = x) \\
& \wedge \forall x \forall y \; (\mathrm{op}(\mathrm{push}(x,y) = y) \\
& \wedge \forall x \; (\neg \mathrm{IsEmpty}(x) \rightarrow \\
& \qquad \mathrm{push}(\mathrm{top}(x), \mathrm{pop}(x)) = x))
\end{aligned}
$$