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Yasuhara Ch. 9 (Propositional Calculus)

Lemma 9.1 (1) - alternate proof

1. $(A \supset B), (B \supset C), A \vdash A$ hypothesis
2. " " " $\vdash (A \supset B)$ "
3. " " " $\vdash B$ m.p. on 1 and 2
4. " " " $\nvdash (B \supset C)$ m.p. on 3 and 4
5. " " " $\vdash C$

6. $(A \supset B), (B \supset C) \vdash (A \supset C)$ Deduction Theorem on 5

7 $(A \supset B) \vdash ((B \supset C) \supset (A \supset C))$ 4 2

8 $\vdash ((A \supset B) \supset ((B \supset C) \supset (A \supset C)))$ Ded Thm

Proof of Lemma 9.1 (3): $\vdash (\sim B \supset (B \supset C))$

1. $\sim B \vdash \sim B$ hypothesis
2. $\sim B \vdash (\sim B \supset (\sim C \supset \sim B))$ axiom 1
3. $\sim B \vdash ((\sim C \supset \sim B) \supset (B \supset C))$ axiom 3
4. $\sim B \vdash (\sim C \supset \sim B)$ m.p. on 1, 2

5. $\sim B \vdash (B \supset C)$ m.p. on 4, 3

6. $\vdash (\sim B \supset (B \supset C))$ Deduction theorem

The truth table for implication

p_1	p_2	$(p_1 \supset p_2)$	$(\sim p_1 \vee p_2)$	$\sim p_1$
t	t	t	t	f
t	f	f	f	f
f	t	t	t	t
f	f	t	t	t

$(A \supset (B \supset A))$	A	B	$(B \supset A)$
t	t	t	t
t	t	f	t
t	f	t	f
t	f	f	t

This show that axiom 1 is a tautology.

Axiom 2 and axiom 3 are tautologies also.

If A is a tautology and $(A \supset B)$ is a tautology, then B is a tautology.

If A is a tautology and $(\sim A \vee B)$ is a tautology, then B is a tautology

Theorem 9.2 If $\vdash_{P_0} A$, then A is a tautology.

Proof By complete induction in the length of the derivation (proof) of A.

Basis. (length 1). Let D be a theorem (of P_0) with a proof of length 1. So, D is an axiom. By exercise 9.9(c), D is a tautology.

Inductive step. Let B be a theorem with a proof of length $k > 1$. If B is an axiom, then the argument of the basis case still holds. If B is not an axiom, then B follows from previous formulas in the derivation using modus ponens. The previous formulas have the form

A and $(A \supset B)$. By exercise 9.9(d), B is a tautology. $\square$

Exercise 9.14

$L(Y) = L(P_0)$ language

$((A \supset B) \supset (A \supset A))$ axiom

$\{A, (A \supset B)\} \to B$ rule of inference (mp)

(a) The axiom is a tautology, as one can check by TT!

A	B	$A \supset B$	$A \supset A$	$((A \supset B) \supset (A \supset A))$
f	f	t	t	t
f	t	t	t	t
t	f	f	t	t
t	t	t	t	t

mp preserves tautologies as shown in Exercise 9.9 (d).
So, yes

(b) 1. $\vdash ((A \supset B) \supset (A \supset A))$ axiom

2. $\vdash ((A \supset B) \supset (A \supset A)) \supset ((A \supset B) \supset (A \supset B))$ axiom with $A \leftarrow A \supset B$, $B \leftarrow A \supset A$

3. $\vdash ((A \supset B) \supset (A \supset B))$ mp on 1, 2

Case 1 Let $\bar{A}$ be an axiom and $(\bar{A} \supset \bar{B})$ be also an axiom

$$\{\bar{A}, (\bar{A} \supset \bar{B})\} \to \bar{B} \quad (A \supset B) \supset (A \supset B))$$

$$((A \supset B) \supset (A \supset A)) \supset (A \supset B) \supset (A \supset B)$$

(mp)

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Exercise 9.6.

The converse of the deduction theorem is:

If $B_1, \ldots B_{k-1} \vdash_{P_0} (B_k \supset C)$ then

$B_1, \ldots, B_{k-1}, B_k \vdash C$.

Proof;

1 $B_1, \ldots, B_{k-1} \vdash_{P_0} (B_k \supset C)$ given

2 $B_1, \ldots, B_{k-1}, B_k \vdash_{P_0} (B_k \supset C)$ defn of derivation relative to hypothesis

Comment: This is a formal way of describing what people mean by "the propositional calculus is monotonic".

3 $B_1, \ldots, B_{k-1}, B_k \vdash_{P_0} B_k$ hypothesis

4 $B_1, \ldots, B_{k-1}, B_k \vdash_{P_0} C$ m.p. on 2, 3

$\square$

Recall Theorem 9.2; If $\vdash_{P_0} A$, then A is a tautology. (The soundness of the propositional calculus; the propositional calculus is sound.)

Theorem 9.5. If $A \in F(P_0)$ and A is a tautology, then $\vdash_{P_0} A$. (The propositional calculus is complete.)

Lemma 9.2 Let $A \in F(P_0)$ and let $p_1, \ldots p_k$ be the propositional variables that occur in A. Consider each row of the t.t. for A and for each p_i write B_i as follows; if p_i is t, then $B_i = p_i$; otherwise, $B_i = \sim p_i$. Similarly, let A' be A if A is t in that row of the tt and let A' be $\sim A$ otherwise.

Then, $B_1, \ldots B_k \vdash_{P_0} A'$

Example of the construction:

$$A = (p_1 \supset (p_2 \supset p_1))$$

$p_1, p_2 \vdash_{P_0} (p_1 \supset (p_2 \supset p_1))$

$p_1, \sim p_2 \vdash_{P_0} (p_1 \supset (p_2 \supset p_1))$

$\sim p_1, p_2 \vdash_{P_0} (p_1 \supset (p_2 \supset p_1))$

$\sim p_1, \sim p_2 \vdash_{P_0} (p_1 \supset (p_2 \supset p_1))$

p_1	p_2	$(p_2 \supset p_1)$	$(p_1 \supset (p_2 \supset p_1))$
t	t	t	t
t	f	t	t
f	t	f	t
f	f	t	t

Example: $A = (p_2 \supset p_1)$

$p_1, p_2 \vdash (p_2 \supset p_1)$

$p_1, \sim p_2 \vdash (p_2 \supset p_1)$

$\sim p_1, p_2 \vdash \sim (p_2 \supset p_1)$

$\sim p_1, \sim p_2 \vdash (p_2 \supset p_1)$

p_1	p_2	$p_2 \supset p_1$
t	t	t
t	f	t
f	t	f
f	f	t

Proof (by induction on the number of connectives) (Note: let n be the number of propositional variables in A)

Basis $(n = 0)$

The formula has the form p_i $(= A)$

On it:

$p_i \vdash p_i$

$\neg p_i \vdash \neg p_i$ ✓

p_i	$A\ (= p_i)$
t	t
f	f

Inductive case. At least one propositional connective. Consider two cases:

A has the form $\sim C$ — see book

A has the form $(B \supset C)$ — see book for beginning — (a)

(b) suppose that A is assigned t, B is assigned t, and C is assigned f

By inductive assumption:

(1) $B_1, \ldots, B_k \vdash B$

(2) $B_1, \ldots, B_k \vdash C$

(3) $B_1, \ldots, B_k \vdash (C \supset (B \supset C))$ axiom 1

(4) $B_1, \ldots, B_k \vdash (B \supset C)$ mp on 2, 3

(c) A is assigned t, B is assigned f, and C is assigned t. Then:

(1) $B_1, \ldots, B_k \vdash \sim B$

(2) $B_1, \ldots, B_k \vdash C$

} ind. assumption

(3) $B_1, \ldots, B_k \vdash (C \supset (B \supset C))$ axiom 1

(4) $B_1, \ldots, B_k \vdash (B \supset C)$ mp on 2, 3

(d) A is t, B is f, C is f.

(1) $B_1, \dots B_k \vdash \sim B$

(2) $B_1, \dots, B_k \vdash \sim C$

$\Big\}$ ind. assumption

(3) $B_1, \dots, B_k \vdash (\sim B \supset (B \supset C))$ lemma 9.1. (3)

(4) $B_1, \dots, B_k \vdash (B \supset C)$ m.p. on (1) and (3)

done with Lemma. 9.2

Proof of Theorem 9.5 (If $A \in P_0$ and A is a tautology, then $\vdash_{P_0} A$.)

If A is a tautology, then it is assigned T in every row of its truth table.

Let $p_1, \dots, p_k$ be the prop. vars of A.

The truth table of A has 2^k rows.

In half of them, p_k is assigned T, so

B_k (of Lemma 9.2) is p_k, and

(1) $B_1, \ldots B_{k-1}, p_k \vdash A$ (by Lemma 9.2)

In the other half of the rows of the tt, p_k is f

(2) $B_1, \ldots B_{k-1}, \neg p_k \vdash A$ (by Lemma 9.2)

(3) $B_1, \ldots, B_{k-1} \vdash (p_k \supset A)$ ded thm on (1)

(4) $B_1, \ldots, B_{k-1} \vdash (\neg p_k \supset A)$ ded thm on (2)

(5) $B_1, \ldots, B_{k-1} \vdash ((p_k \supset A) \supset ((\sim p_k \supset A) \supset A))$

Lemma 9.1 (8)

Use m. p. twice (5, 3, 4) ;

(6) $B_1, \ldots, B_{k-1} \vdash A$

Do this (1–6) $k-1$ more times, and obtain,

$$\vdash A$$

$\square$