781 2011-01-20

Note Title 2011-01-20

Yasuhara Ch. 9 (Propositional Calculus)

Lemma 9.1 (1) - alternate proof

1. $(A \supset B), (B \supset C), A \vdash A$ hypothesis

2. " " " $\vdash (A \supset B)$ "

3. " " " $\vdash B$ m.p. on 1 and 2

4. " " " $\not\vdash (B \supset C)$ m.p. on 3 and 4

5. " " " $\vdash C$

6. $(A \supset B), (B \supset C) \vdash (A \supset C)$ Deduction Theorem on 5

7 $(A \supset B) \vdash ((B \supset C) \supset (A \supset C))$ 4 2

8 $\vdash ((A \supset B) \supset ((B \supset C) \supset (A \supset C)))$ Ded Thm

Proof of Lemma 9.1 (3): $\vdash (\sim B \supset (B \supset C))$

1. $\sim B \vdash \sim B$ hypothesis
2. $\sim B \vdash (\sim B \supset (\sim C \supset \sim B))$ axiom 1
3. $\sim B \vdash ((\sim C \supset \sim B) \supset (B \supset C))$ axiom 3
4. $\sim B \vdash (\sim C \supset \sim B)$ m.p. on 1, 2

5. $\sim B \quad \vdash (B \supset C)$ m.p. on 4, 3

6. $\quad \vdash (\sim B \supset (B \supset C))$ Deduction theorem

The truth table for implication

p_1	p_2	$(p_1 \supset p_2)$	$(\sim p_1 \vee p_2)$	$\sim p_1$
t	t	t	t	f
t	f	f	f	f
f	t	t	t	t
f	f	t	t	t

$(A \supset (B \supset A))$	A	B	$(B \supset A)$
t	t	t	t
t	t	f	t
t	f	t	f
t	f	f	t

This show that axiom 1 is a tautology.

Axiom 2 and axiom 3 are tautologies also.

If A is a tautology and $(A \supset B)$ is a tautology, then B is a tautology.

If A is a tautology and $(\sim A \vee B)$ is a tautology, then B is a tautology

Theorem 9.2 If $\vdash_{P_0} A$, then A is a tautology.

Proof By complete induction in the length of the derivation (proof) of A.

Basis. (length 1). Let D be a theorem (of P_0) with a proof of length 1, So, D is an axiom By exercise 9.9 (c), D is a tautology.

Inductive step. Let B be a theorem with a proof of length $k > 1$. If B is an axiom, then the argument of the basis case still holds. If B is not an axiom, then B follows from previous formulas in the derivation using modus ponens. The previous formulas have the form

A and $(A \supset B)$. By exercise 9.9 (d), B is a tautology. $\square$

Exercise 9.14

$L(Y) = L(P_0)$ language

$((A \supset B) \supset (A \supset A))$ axiom

$\{A, (A \supset B)\} \to B$ rule of inference (mp)

(a) The axiom is a tautology, as one can check by TT!

A	B	$A \supset B$	$A \supset A$	$((A \supset B) \supset (A \supset A))$
f	f	t	t	t
f	t	t	t	t
t	f	f	t	t
t	t	t	t	t

mp preserves tautologies as
shown in Exercise 9.9 (d).
So, yes

(b) 1. $\vdash ((A \supset B) \supset (A \supset A))$ axiom

2. $\vdash ((A \supset B) \supset (A \supset A)) \supset ((A \supset B) \supset (A \supset B))$ axiom with $A \leftarrow A \supset B$, $B \leftarrow A \supset A$

3. $\vdash ((A \supset B) \supset (A \supset B))$ mp on 1, 2

Case 1 Let $\bar{A}$ be an axiom and $(\bar{A} \supset \bar{B})$ be also an axiom

$$\{\bar{A}, (\bar{A} \supset \bar{B})\} \to \bar{B} \quad ((A \supset B) \supset (A \supset B)) \quad (mp)$$

$$((A \supset B) \supset (A \supset A)) \supset (A \supset B) \supset (A \supset B)$$