UNIVERSITY OF SOUTH CAROLINA

ECHE 460 CHEMICAL ENGINEERING LABORATORY I

Heat Transfer Analysis in Solids

Prepared by:

M. L. Price, and Professors M. A. Matthews and T. Papathansiou

Department of Chemical Engineering
Swearingen Engineering Center
University of South Carolina
Columbia, SC 29208

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Objectives

In this project, you will gather unsteady-state data from the conductive heating of solid shapes that are made of various materials. The solids are being heated (or cooled) in a water bath so the process involves a convective boundary condition. Therefore, the heat transfer process will be modeled as transient heat conduction with a convective boundary. The important variables to be considered are the shape (geometry) of the solid and the physical properties (type of material, density, specific heat, and thermal conductivity) of the solid. The heating process will be modeled with varying levels of complexity. Depending on the particular experiment and amount of known information, these experiments can be used to determine either (a) thermophysical properties of the solid (specific heat or thermal conductivity) or (b) the convective heat transfer coefficient at the solid/fluid interface. Students will formulate more specific objectives in keeping with their experimental plans.

Introduction

Heat transfer is relevant to all engineering disciplines, and is a fascinating part of the engineering sciences. Heat transfer phenomena play an important role in many industrial and environmental problems. As an example, consider the vital area of energy production and conversion. There is not a single application in this area that does not involve heat transfer effects in some way. In the generation of electrical power, whether through nuclear fission or fusion, the combustion of fossil fuels, or the use of geothermal energy sources, there are numerous heat transfer problems that must be solved. These problems involve conduction, convection, and radiation processes and relate to the design of systems such as boilers, condensers, and turbines. One is often confronted with the need to maximize heat transfer rates and to maintain the integrity of materials in high-temperature environments.

Experimental Procedure:

In the laboratory, we have a number of solid shapes (cylinders, spheres, rectangles) that have been fabricated from a number of different materials (copper, brass, aluminum, stainless steel, Teflon™, and possibly others). Each shape has been fitted with a thermocouple (electrical temperature sensor) that is located at the precise center of the shape. You should make a complete inventory table of all available shapes and materials. You will select from among these to get sufficient data to meet your objectives. If the heat transfer coefficient for a particular geometry and set of fluid conditions is known, you can use the experimental results to compute the heat capacity and thermal conductivity of the solid. Conversely, if the thermal conductivity and heat capacity of the solid are known, you can use the results to compute the heat transfer coefficient for a given geometry. The raw data are centerline temperature versus time for a given solid shape. In addition, you must record all relevant descriptive data for the shape that are being used. Read all directions and examine the required calculations before beginning experimentation.

The experiments themselves are simple. After recording the initial temperature of each shape, the solid will be immersed in a constant-temperature bath and the transient temperature response will be recorded. It takes approximately one hour to establish the setpoint temperature for the water bath, so must coordinate with the teaching assistant for one of you to get started early to establish the bath temperature. Only one bath temperature can be set each lab period. Generally speaking, each group will do one set of runs with heating, and another set of runs during the second lab period with cooling. You will note that the water bath has a stirrer. By turning off the stirrer, you can estimate the heat transfer coefficient under free convection conditions.

Step-by-step instructions are given below. Consult closely with your teaching assistant for further details.

1. Prior to the 2:30 starting time, insure that the water bath is properly filled. The water should be one inch from the top and should cover the heater, pump, and one inch of the temperature sensor.

2. Prior to the 2:30 starting time, turn on the water bath, controller, and potentiometer. Set the controller to the desired water bath temperature (the *setpoint* temperature).

    a. In order to set the controller to the desired temperature, press the Set button on the controller. Then enter the value of the set point, including all decimal places. Then press enter. The setpoint will then appear on the controller screen.

    b. Allow sufficient time for the water bath to stabilize completely at the setpoint. It may take over an hour. It will take longer to cool the water than to heat it.

    c. The spare thermocouple should be placed in the small hole next to the controller and plugged into the outlet labeled 1. Then the measured value recorded by the thermocouple and associated equipment can be compared against the controller value.

3. Turn on the computer and monitor. After Windows is loaded, double click on the icon labeled "Heat Conduction" to start the monitoring program, which is the software LabView. A window will appear which will specify the username as Unit Operations Laboratory. Click OK or press Enter.

4. The first day of experimentation will be to heat all the solids from room temperature to a higher temperature. The bath temperature will be between 40 and 80 ^{o}C. The particular temperatures for your group will be given by the TA. The second day the solids will be cooled from room temperature to a lower temperature between 5 and 15 ^{o}C. The temperature range of the water bath is from 3 ^{o}C to 90 ^{o}C. Please be very careful not to burn yourself with hot water or the hot solids. Lift the solids using the support wire, and use gloves or tongs if needed.

5. Record the initial temperature (room temperature) of the solids. Prepare to monitor the thermocouple temperature by using the mouse to click on the start or run arrow at the top of the Heat Conduction program window. It is simplest to start a new data file for each experiment. See 6 below for more details on the data acquisition software.

6. You need to store the data file name in the appropriate space on the hard drive. Be sure to use the following form when specifying the file name in the data acquisition program:

    C:\Heat Conduction*yourfilename*.txt

    a. When you are ready to begin recording data to the file, use the mouse to toggle the "off" key by "Enable Filing". You should not begin recording until just before the solid is placed in the bath.

b. The computer program will record the temperature and time elapsed since the monitoring was begun. In order to determine the relative time since the beginning of the testing of the particular solid, one will need to adjust the time values.

7. After the bath temperature is stabilized and the data acquisition system is ready, you may immerse the solid in the water bath. First, hook the solid to the fish-eye on the frame. Insure that you have begun recording the data. Then lower the solid into the water bath until completely submerged. Don't drop the solid in the bath, but do try to get it immersed quickly. At the instant the solid is immersed, note the time on the data acquisition system. There may be a lag between the time shown on the data acquisition and the actual "time zero" that the solid began heating.
8. When the solid has reached the temperature of the bath and remains stable, disable the data acquisition. Remove the solid and hang it back over the plastic tub so that it can cool to room temperature. Make sure that the data are saved on the hard disk, and on a floppy for backup.
9. Repeat the items 6 through 8 for each solid.
10. Repeat some experiments as time allows. In the meantime, immerse the solids in a bucket of room-temperature water in order to hasten the return to room temperature.
11. Repeat steps 5-9 for the second setpoint temperature.

Theoretical Background (taken from the text by Incropera and DeWitt)

Rate Law for Heat Conduction

For heat conduction, the rate equation is known as Fourier's law. For the one-dimensional plane wall shown, having a temperature distribution T(x), Fourier's law is

$$q''_x = - k\, dT/dx \qquad (1)$$

The heat flux q''_x (W/m^2) is the heat transfer rate in the x direction per unit area perpendicular to the direction of transfer, and it is proportional to the temperature gradient, dT/dx, in this direction. The proportionality constant k is a transport property known as the thermal conductivity (W/mK) and is a thermophysical property of the solid material. The minus sign is a consequence of the fact that heat is transferred in the direction of decreasing temperature. Note that this equation provides a heat flux, that is, the rate of heat transfer per unit area. The heat rate by conduction, q_x (W), through a plane wall of area A is then the product of the flux and the area,

$$q_x = q''_x * A. \quad (2)$$

Rate Law for Heat Convection

The convective heat transfer mode is comprised of two mechanisms. In addition to energy transfer due to random molecular motion (diffusion), energy is also transferred by the bulk, or macroscopic, motion of the fluid. The rate of convective heat transfer is enhanced by vigorous stirring or rapid flow. We are especially interested in convection heat transfer that occurs between a fluid in motion and a bounding surface when the two are at different temperatures.

For convection, the rate equation is of the form

$$q'' = h(T_s - T_\infty) \qquad (3)$$

where q'', the convective heat flux (W/m^2), is proportional to the difference between the surface temperature T_s and the bulk fluid temperature T_∞. Equation (3) is known as Newton's law of cooling, and the proportionality constant h (W/m^2*K) is termed the convection heat transfer coefficient. In contrast to thermal conductivity k, h is not a thermophysical property of a material; h depends on conditions of agitation, stirring, and the local velocity in the boundary layer between the heating fluid and the solid surface. h is influenced by surface geometry, the nature of the fluid motion, and various transport and thermophysical properties of the heat transfer fluid, including its viscosity, thermal conductivity, heat capacity, and density.

When Equation (3) is used, the convection heat flux is presumed to be positive if heat is transferred from the surface ($T_s > T_\infty$) and negative if heat is transferred to the surface ($T_\infty > T_s$). However, if $T_\infty > T_s$, there is nothing to preclude us from expressing Newton's law of cooling as

$$q'' = h(T_\infty - T_s) \qquad (4)$$

in which case heat transfer is positive if it is to the surface.

Transient Heat Conduction

We recognize that many heat transfer problems are time dependent. Such *unsteady state*, or *transient*, problems typically arise when the boundary conditions of a system are changed. For example, if the surface temperature of a solid is altered, the temperature at each point in the solid will also begin to change. The changes will continue to occur until a *steady-state* temperature distribution is reached. Consider a hot metal forging that is removed from a furnace and exposed to a cool air stream. Energy is transferred by convection and radiation from its surface to the surrounding air. Energy transfer by conduction occurs from the interior of the metal to the surface, and the temperature at each point in the billet decreases until a steady-state condition is reached. Such time-dependent effects occur in many industrial heating and cooling processes.

To determine the time dependence of the temperature distribution within a solid during a transient process, we begin by solving the appropriate energy balance and rate law. The precise form depends on the geometry of the solid (plane wall, cylinder, sphere). In addition, the boundary conditions must be specified. In the following discussion we will present two general approaches to solving transient heat transfer in a uniform solid: the Lumped Capacitance method, and the more rigorous method of solving Fourier's law that includes spatial temperature gradients.

The Lumped Capacitance Method

A common transient conduction problem is one in which a solid experiences a sudden change in its thermal environment. Consider a hot metal forging that is initially at a uniform temperature T_i and is quenched by immersing it in a liquid of lower temperature $T_\infty < T_i$. If the quenching is said to begin at time t = 0, the temperature of the solid will decrease for time $t > 0$, until it eventually reaches T_∞.

The essence of the lumped capacitance method is the assumption that the temperature of the solid is *spatially uniform* at any instant during the transient process. This assumption means that temperature gradients within the solid are negligible (*dT/dx = 0)*. This is equivalent to saying that the rate of conduction of heat inside the solid is very large compared to the rate of heat convection at the fluid/solid interface.

By examining Fourier's law, eq. (1), we see that heat conduction in the absence of a temperature gradient implies infinitely large thermal conductivity. Such a condition is clearly impossible. However, although the condition is never satisfied exactly, it is closely approximated if the resistance to conduction within the solid is small compared with the resistance to heat transfer between the solid and its surroundings.

By neglecting temperature gradients within the solid, we no longer need so solve the partial differential equations (spatial dependence) of the problem that are part of Fourier's law of conduction. Instead, the transient temperature response is determined simply by formulating an overall energy balance on the solid. This balance relates the rate of heat transfer at the solid surface (Newton's law) to the rate of change of the internal energy (*U*) of the solid.

$$-hA_s(T-T_\infty) = dU/dt = \rho Vc\, dT/dt \qquad (5)$$

In equation (5), the solid has volume V, surface area A_s, temperature T, density ρ, and specific heat c. The assumptions behind this analysis mean that the surface temperature T is equal to the interior temperature T everywhere throughout the solid. Introducing the temperature driving force θ

$$\theta \equiv T - T_\infty \qquad (6)$$

and recognizing that $(d\theta/dt) = (dT/dt)$, it follows that

$$\frac{\rho Vc}{hA_s}\frac{d\theta}{dt} = -\theta \qquad (7)$$

Separating variables and integrating from the initial condition, for which $t = 0$ and $T(0) = T_i$, we then obtain

$$\frac{\rho Vc}{hA_s}\int_{\theta i}^{\theta}\frac{d\theta}{\theta} = -\int_0^t dt \qquad (8)$$

where

$$\theta_i \equiv T_i - T_\infty \qquad (9)$$

Evaluating the integrals it follows that

$$\frac{\rho Vc}{hA_s}\ln\frac{\theta_i}{\theta} = t \qquad (10)$$

or

$$\frac{\theta}{\theta i} = \frac{T-T_\infty}{T_i-T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho Vc}\right)t\right] \qquad (11)$$

If the solid properties and h, are known, Equation (10) may be used to determine the time required for the solid to reach some temperature T. Conversely, Equation (11) may be used to compute the temperature reached by the solid at some time t. Finally, if the solid properties and $T(t)$ are known, equation (10) can be used to evaluate h. Thus, experimenters must understand what is known and what is unknown so that appropriate use of the data is determined.

The foregoing results indicate that the difference between the solid and fluid temperatures must decay exponentially to zero as t approaches infinity. From Equation (11) it is also evident that the quantity ($\rho Vc/hA_s$) may be interpreted as a *thermal time constant*. This time constant may be expressed as

$$\tau_t = \left(\frac{1}{hA_s}\right)(\rho Vc) = R_t C_t \qquad (12)$$

where R_t is the resistance to convection heat transfer and C_t is the *lumped thermal capacitance* of the solid. Any increase in R_t or C_t will cause a solid to respond more slowly to changes in its thermal environment and will increase the time required to reach thermal equilibrium ($\theta = 0$). This behavior is analogous to the voltage decay that occurs when a capacitor is discharged through a resistor in an electrical RC circuit.

Validity of the Lumped Capacitance Method

From the foregoing results it is easy to see why there is a strong preference for using the lumped capacitance method. It is certainly the simplest and most convenient method that can be used to solve transient conduction problems. Hence it is important to determine under what conditions it may be used with reasonable accuracy.

To develop a suitable criterion, we will briefly consider steady-state conduction through the plane wall of area A. (Although we will consider steady-state conditions, this criterion is readily extended to transient processes.) One surface is maintained at a temperature $T_{s,1}$ and the other surface is exposed to a fluid of temperature $T_\infty < T_{s,1}$. The temperature of this surface will be some intermediate value , $T_{s,2}$, for which $T_\infty < T_{s,2} < T_{s,1}$. Hence, under steady-state conditions the surface energy balance shows that heat transfer through the plane wall by conduction must equal heat transfer by convection at the wall/fluid boundary:

$$(kA/L)\ (T_{s,1} - T_{s,2}) = hA\ (T_{s,2} - T_\infty) \qquad (13)$$

Rearranging, we then obtain

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{cond}}{R_{conv}} = \frac{hL}{k} \equiv Bi \qquad (14)$$

The quantity h*L / k appearing in this equation is a *dimensionless group* called the *Biot number*, *Bi*. *Bi* plays a fundamental role in conduction problems that involve surface convection effects. According to Equation (14), the Biot number provides a measure of the temperature drop across the wall relative to the temperature difference between the wall surface and the fluid. Note especially the case where $Bi \ll 1$. For this case, it is reasonable to *assume* a uniform temperature distribution across a solid at any time during a transient process. This result may also be associated with interpretation of the Biot number as a ratio of thermal resistances, Equation 14. If $Bi \ll 1$, *the resistance to conduction within the solid is much less than the resistance to* *convection across the fluid boundary layer*. Hence the assumption of a uniform temperature distribution inside the solid is reasonable.

We have introduced the Biot number because of its significance to transient conduction problems. Consider a plane wall that is initially at a uniform temperature T_i and experiences convection cooling when it is immersed in a fluid of $T_\infty < T_i$. The problem may be treated as one-dimensional in x, and we are interested in the temperature variation with position and time, T(x,t). This variation is a strong function of the Biot number, and three conditions are possible. For Bi << 1 the temperature gradient in the solid is small and $T(x,t) \approx T(t)$. Virtually all the temperature difference is between the solid and the fluid, and the solid temperature remains nearly uniform as it decreases to T_∞. For moderate to large values of the Biot number, however, the temperature gradients within the solid are significant. Hence T = T(x,t). Note that for Bi >> 1, the temperature difference between the two edges of the solid is now much larger than that between the surface and the fluid.

It is important to emphasize the importance of the lumped capacitance method. Its inherent simplicity renders it the preferred method for solving transient conduction problems. Hence, when confronted with such a problem, *the very first thing that one should do is calculate (or estimate) the Biot number*. If the following condition is satisfied,

$$Bi = \frac{hL_c}{k} < 0.1 \qquad (15)$$

then the error associated with using the lumped capacitance method is small. For convenience, it is customary to define the *characteristic length* of equation (15) as the ratio of the solid's volume to surface area, $L_c \equiv V/A_s$. Such a definition facilitates calculation of L_c for solids of complicated shape and reduces to the half-thickness L for a plane wall of thickness 2L, to $r_o/2$ for a long cylinder, and to $r_o/3$ for a sphere. However, if one wishes to implement the criterion in a conservative fashion, L_c should be associated with the length scale corresponding to the maximum spatial temperature difference. Accordingly, for a symmetrically heated (or cooled) plane wall of thickness 2L, L_c would remain equal to the half-thickness L. However, for a long cylinder or sphere, L_c would equal the actual radius r_o, rather than $r_o/2$ or $r_o/3$.

Finally, we note that, with $L_c \equiv V/A_s$, the exponent of Equation 11 may be expressed as

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2} \qquad (16)$$

or

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo \qquad (17)$$

where

$$Fo \equiv \frac{\alpha t}{L_c^2} \qquad (18)$$

is termed the Fourier number. It is a *dimensionless time*, which, with the Biot number, characterizes transient conduction problems. Substituting equation (18) into (11), we obtain

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo) \qquad (19)$$

Beyond the Lumped Capacitance Method: Rigorous Consideration of Spatial Gradients

Situations frequently arise for which we must cope with the fact that spatial gradients within the solid medium are non-negligible. (This is equivalent to saying that the Biot number is sufficiently large that the lumped capacitance method cannot be applied.) In their most general form, transient conduction problems are described by the heat equation (Fourier's law) in rectangular, cylindrical, or spherical coordinates. The solution to these partial differential equations provides the variation of temperature with both time and the spatial coordinates. However, in many problems, such as the plane wall or the sphere, only one spatial coordinate is needed to describe the internal temperature distribution.

Consider first a one-dimensional plane wall (a rectangle with heat conduction in only the *x*-direction). Also assume that the thermal conductivity is constant. An energy balance relating the rate of change of internal energy to the rate of heat conduction reduces to Equation 20:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (20)$$

In this equation we have introduced the thermal diffusivity α, which has units of m^2/s:

$$\alpha = k / (\rho\, C_p) \qquad (21)$$

It measures the ability of a material to conduct thermal energy relative to its ability to store thermal energy. Materials with large α will have a rapid internal temperature response to changes in their thermal environment, while materials with small α will respond more sluggishly, that is, the interior temperature will take longer to reach a new equilibrium condition.

To solve Equation (20) for the temperature distribution $T(x,t)$, it is necessary to specify an *initial* condition and two *boundary conditions*. We consider the symmetrical plane wall of thickness $2L$, with a centerline at $x=0$ (Figure 1).

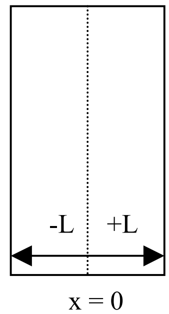

Figure 1: The 1-D Symmetrical Plane Wall

For the typical transient conduction problem, the initial condition is

$$T(x,0) = T_i \qquad (22)$$

and the boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad (23)$$

and

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=+L;-L} = h[T(L,t) - T_\infty] \qquad (24)$$

Equation (22) shows a uniform temperature distribution at time t = 0; Equation (23) reflects the *symmetry requirement* for the midplane of the wall; and Equation (24) describes the rate of conductive and convective heat transfer at the surface for time $t>0$. From Equations (21) to (24), it is evident that, in addition to depending on x and t, temperatures in the wall also depend on a number of physical parameters. In particular

$$T = T(x, t, T_i, T_\infty, L, k, \alpha, h) \qquad (25)$$

The foregoing problem may be solved analytically or numerically. It is important to note the advantages that may be obtained by *nondimensionalizing* the governing equations. This may be done by arranging the relevant variables into suitable *groups*. Consider the dependent variable T. If the temperature difference $\theta \equiv (T - T_\infty)$ is divided by the *maximum possible temperature difference* $\theta_i \equiv (T_i - T_\infty)$, a dimensionless form of the dependent variable may be defined as

$$\theta^* \equiv \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} \qquad (26)$$

Accordingly, θ^* must lie in the range $0 \le \theta^* \le 1$. A dimensionless spatial coordinate may be defined as

$$x^* \equiv \frac{x}{L} \qquad (27)$$

where L is the half-thickness of the plane wall, and a dimensionless time may be defined as

$$t^* \equiv \frac{\alpha t}{L^2} \equiv Fo \qquad (28)$$

where t* is equivalent to the dimensionless *Fourier number*, Equation (18).

Substituting the definitions of Equations (26) to (28) into Equations (20) to (24), the heat equation becomes

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo} \qquad (29)$$

and the initial and boundary conditions become

$$\theta^*(x^*,0) = 1 \qquad (30)$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0} = 0 \qquad (31)$$

and

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1} = -Bi\,\theta^*(1,t^*) \qquad (32)$$

where the Biot number is $Bi = hL/k$. In dimensionless form the functional dependence may now be expressed as

$$\theta^* = f(x^*, Fo, Bi) \qquad (33)$$

Comparing Equations (25) and (33), the considerable advantage associated with casting the problem in dimensionless form becomes apparent. Equation (33) implies that *for a prescribed geometry, the transient temperature distribution is a universal function of* x*, *Fo, and Bi*. That is, the *dimensionless solution* assumes a prescribed form that does not depend on the particular value of T_i, T_∞, L, k, α, or h. Since this generalization greatly simplifies the presentation and utilization of transient solutions, the dimensionless variables are used extensively.

Geometry #1: The 1-D plane wall with convection at the boundary

Exact, analytical solutions to transient conduction problems have been obtained for many simplified geometries and boundary conditions and are well documented. Several mathematical techniques, including the method of separation of variables, may be used for this purpose, and typically the solution for the dimensionless temperature distribution, Equation (33), is in the form of an infinite series. However, except for very small values of the Fourier number, infinite series may be approximated by a single term and the results may be represented in a convenient graphical form.

Exact Solution

Consider the plane wall of thickness $2L$. If the thickness is small relative to the width and height of the wall, it is reasonable to assume that conduction occurs exclusively in the x

direction. If the wall is initially at a uniform temperature, $T(x,0) = T_i$, and is suddenly immersed in a fluid of $T_\infty \neq T_i$, the resulting temperatures may be obtained by solving Equation (29) subject to the conditions of Equations (30) to (32). Since the convection conditions for the surfaces at $x^* = \pm 1$ are the same, the temperature distribution at any instant must be symmetrical about the midplane ($x^* = 0$). An exact solution to this problem has been obtained and is of the form

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo)\cos(\zeta_n x^*) \qquad (34)$$

where $Fo = \alpha t/L^2$ and the coefficient C_n is

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin(2\zeta_n)} \qquad (35)$$

and the discrete values (*eigenvalues*) of ζ_n are positive roots of the transcendental equation

$$\zeta_n \tan \zeta_n = Bi \qquad (36)$$

The first four roots of this equation are given in Appendix B.3 of the textbook by Incropera and De Witt.

Approximate Solution

It can be shown that for values of $Fo > 0.2$, the infinite series solution, Equation (34), can be approximated by the first term of the series. Invoking this approximation, the dimensionless form of the temperature distribution becomes

$$\theta^* = C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*) \qquad (37)$$

or

$$\theta^* = \theta_o^* \cos(\zeta_1 x^*) \qquad (38)$$

where $\theta_o^* \equiv (T_o - T_\infty) / (T_i - T_\infty)$ represents the midplane ($x^* = 0$) temperature

$$\theta_o^* = C_1 \exp(-\zeta_1^2 Fo) \qquad (39)$$

An important implication of Equation (38) is that *the time dependence of the temperature at any location within the wall is the same as that of the midplane temperature*. The coefficients C_1 and ζ_1 are evaluated from Equations (35) and (36), respectively, and are given in Table 5.1 of Incropera and De Witt for a range of Biot numbers. Graphical representations of the one-term approximations have been developed and are presented in Appendix D. Although the associated charts provide a convenient means of solving one-dimensional transient conduction problems for Fo > 0.2, better accuracy may be obtained by using the equations.

Geometry 2 and 3: 1-D radial cylinder or sphere with convection

For an infinite cylinder (or sphere) of radius r_o that is at an initial uniform temperature, an exact series solution may be obtained for the time dependence of the radial temperature distribution. Furthermore, a one-term approximation may be used for most conditions. The infinite cylinder is an idealization that permits the assumption of one-dimensional conduction in the radial direction. It is a reasonable approximation for cylinders having $L/r_o \geq 10$.

Infinite Cylinder: Exact Solution

Exact solutions to the transient, one-dimensional form of the heat equation have been developed for the infinite cylinder. For a uniform initial temperature and convective boundary conditions, the solution is as follows.

In dimensionless form, the temperature is

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n r^*) \quad (40)$$

where

$$Fo = \alpha t/2r_o \qquad (41)$$

$$C_n = 2/\zeta_n * J_1(\zeta_n)/ (J_0^2(\zeta_n) + J_1^2(\zeta_n) \qquad (42)$$

and the discrete values of ζ_n are positive roots of the transcendental equation

$$\zeta_n * (J_1(\zeta_n))/(J_0(\zeta_n)) = Bi \qquad (43)$$

The operators J_1 and J_0 are Bessel functions of the first kind and their values are tabulated in Appendix B.4 of Incropera and DeWitt.

Infinite Cylinder: Approximate Solution

For the infinite cylinder, the foregoing series solution can again be approximated by a single term for Fo>0.2. Hence, as for the case of the plane wall, the time dependence of the temperature at any location within the radial system is the same as that of the centerline.

The one-term approximation to Equations (40) through (43) is

$$\theta^* = C_1 \exp(-\zeta^2_1 Fo) J_0(\zeta_1 r^*) \qquad (44)$$

or

$$\theta^* = \theta_o^* J_0(\zeta_1 r^*) \qquad (45)$$

where θ_o* represents the centerline temperature

$$\theta_o* = \frac{T_o - T_\infty}{T_i - T_\infty} \qquad (46)$$

and is of the form

$$\theta_o* = C_1 \exp(-\zeta^2{}_1 Fo) \qquad (47)$$

Values of the coefficients C_1 and ζ_1 have been determined and are listed in Table 5.1 of Incropera and De Witt for a range of Biot numbers.

Sphere: Exact Solutions

Exact solutions to the transient, one-dimensional form of the heat equation have been developed for the sphere. For a uniform initial temperature and convective boundary conditions, the solution is as follows.

$$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n{}^2 Fo)\,(1/(\zeta_n r^*))\,\sin(\zeta_n I) \qquad (48)$$

where

$$Fo = \alpha t / 2r_o$$

$$C_n = \frac{4[\sin(\zeta_n) - \zeta_n \cos(\zeta_n)]}{2\zeta_n - \sin(2\zeta_n)} \qquad (49)$$

and the discrete values of ζ_n are positive roots of the transcendental equation

$$1 - \zeta_n \cot \zeta_n = Bi \qquad (50)$$

Sphere: Approximate Solution

For the sphere, the foregoing series solution can again be approximated by a single term for $Fo > 0.2$. Hence, as for the case of the plane wall, the time dependence of the temperature at any location within the radial system is the same as that of the centerline.

From Equation 59, the one-term approximation is

$$\theta* = C_1 \exp(-\zeta^2{}_1 Fo)\,(1/(\zeta_1 r*))\,\sin(\zeta_1 r*) \qquad (51)$$

or

$$\theta* = \theta_o* \,(1/(\zeta_1 r*))\,\sin(\zeta_1 r*) \qquad (52)$$

where θ_o* represents the center temperature and is of the form

$$\theta_o* = C_1 \exp(-\zeta^2{}_1 Fo) \qquad (53)$$

Values of the coefficients C_1 and ζ_1 have been determined for a range of Biot numbers.

Multidimensional Effects

Transient problems are frequently encountered for which two-and even three-dimensional effects are significant. Solution to a class of such problems can be obtained from the one-dimensional results.

Consider immersing a *short* cylinder that is initially at a uniform temperature T_i, in a fluid of temperature $T_\infty \neq T_i$. Because the length and diameter are comparable, the subsequent transfer of energy by conduction will be significant for both the r and x coordinate directions. The temperature within the cylinder will therefore depend on r, x, and t.

Assuming constant properties and no generation, the appropriate form of the heat equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (54)$$

where x designates the axial coordinate. A closed-form solution to this equation may be obtained by the separation of variables method. Although we will not consider the details of this solution, it is important to note that the end result may be expressed in the following form:

$$\frac{T(r,x,t)-T_\infty}{T_i-T_\infty}=\left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\frac{Plane}{wall}}\cdot\left.\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right|_{\frac{Infinite}{cylinder}} \qquad (55)$$

That is, the two-dimensional solution may be expressed as a *product* of one-dimensional solutions that correspond to those for a plane wall of thickness 2L and an infinite cylinder of radius r_o. For Fo > 0.2, these solutions are provided by the one-term approximationsas well as by graphical solutions that are available in standard texts and reference books.

Results for other multidimensional geometries are available. In each case the multidimensional solution is prescribed in terms of a product involving one or more of the following one-dimensional solutions:

$$S(x,t)\equiv\left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\frac{Semi-infinite}{solid}} \qquad (56)$$

$$P(x,t)\equiv\left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\frac{Plane}{wall}} \qquad (57)$$

$$C(r,t)\equiv\left.\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right|_{\frac{Infinite}{cylinder}} \qquad (58)$$

The x coordinate for the semi-infinite solid is measured from the surface, whereas for the plane wall it is measured from the midplane. In using graphical solutions the coordinate origins should carefully be noted. The transient, three-dimensional temperature distribution in a rectangular parallelpiped, is then, for example, the product of three one-dimensional solutions for plane walls of thicknesses $2L_1$, $2L_2$, and $2L_3$. That is,

$$\frac{T(x_1, x_2, x_3, t) - T_\infty}{T_i - T_\infty} = P(x_1, t) \cdot P(x_2, t) \cdot P(x_3, t) \qquad (59)$$

The distances x_1, x_2, and x_3 are all measured with respect to a rectangular coordinate system whose origin is at the center of the parallelpiped.

Suggested Calculations and Discussion:

Refer to this handout, Chapter 3 and Table 4.10 of the Mills textbook, and to the textbook by Incropera and Dewitt. Generally speaking, you will first analyze the data from the pure component solids (aluminum and copper). Assuming all properties of the metal are known, both the lumped capacitance theory and the more rigorous theory including spatial conduction effects will be used to determine *h*. After examining these results thoroughly, the analysis will proceed to the results with the alloy metals. You will assume that some of the metal alloy properties are unknown, but that the value of *h* determined from aluminum and copper solids will hold for the alloys of the same geometry. You will use the experimental data to determine certain properties of the metal.

The instructions below tell you to "calculate this and plot that." Remember, however, that in the oral presentation and technical report that you should arrange tables and figures that are easy to follow and to discuss. Therefore, you may want to combine certain curves and plots for ease and clarity of presentation. Also, your objectives are not merely to produce certain tables and plots; your discussion must demonstrate critical thinking and depth of analysis.

A. Determination of *h* from transient temperature response in pure solids. For pure solids (aluminum and copper), we will consider all physical properties known and will determine *h*.

<u>Lumped capacitance method</u>

1. Look up the thermal conductivity (k), the density (ρ), and the specific heat (C) of the pure metals. Calculate the thermal diffusivity.
2. Using the data, calculate, plot, and tabulate the dimensionless centerline temperature as a function of time. Remember to adjust the recorded time to the actual time elapsed after the immersion of the solid.
3. Calculate the Fourier number as a function of time.
4. Using the lumped capacitance method, determine the best single experimental value of the convective heat transfer coefficient *h* for each metal and each shape. Compare the values of *h* obtained from each pure solid. If you also obtained data under free convection conditions, use the same analysis to compute h_{free} and compare these to the values of *h* obtained from the stirred (forced convection) experiments.
5. Using the best value of *h*, compute the dimensionless temperature as a function of time and plot the results, comparing to the actual experimental data. How good is this model and the parameter *h*?

6. Considering the Biot number, discuss whether the lumped capacitance approach seems valid to determination of *h*.

Analysis with convection plus internal conduction

1. For each experiment, calculate the Fourier number from the experimental data. Locate the regime where the truncated (one-term) solutions given in this handout are valid.
2. Linearize the one-term solution, and plot the data. Perform linear regression to determine C_1 and ζ_1. For a given shape, how do your values of these two parameters compare with tabulated values? Do you values of C_1 and ζ_1 conform to the analytical relation (*e.g.*, eq. 41).
3. From the value of ζ_1, compute the Biot number and then the heat transfer coefficient *h*. Tabulate *h* for each shape and each type of convection, forced and free.
4. Compare the values of *h* as obtained from the lumped capacitance and spatially-dependent analyses. Discuss.
5. Search the texts and literature for a method of estimating *h,* or for other experimental data. How do your results compare to other measurements or estimates of *h*?

B. For alloys and other materials, we will consider *h* known and will compute unknown physical properties (k or c) of the alloys.

Lumped Capacitance Method

1. Determine or find the density for the alloys (stainless steel and brass) and plastics.
2. Use the best known value of *h* from experimental data on pure solids from part A, and use the lumped capacitance method to determine c (the specific heat) for these materials. Based on what is known from Part A, is this a valid method to determine c? If possible, obtain literature values for c and compare to your experimentally-determined values.

Analysis with convection plus internal conduction

1. Now with the best values of *h* and c, use the spatially-dependent solutions to determine k, the thermal conductivity of the alloys. Do this by determining C_1 and ζ_1 from linear regression of the data, and from ζ_1 determine the Biot number and hence k.

Other points for analysis and discussion:

1. Are end effects important in the cylinder? For situations such as the cylinder, use the method of multidimensional effects to calculate the desired properties.
2. Use correlations or other means to calculate the heat transfer coefficient. What could be done to vary *h*? What can be done to decrease Bi?
3. Compare any values of c or k that you compute to literature (handbook) data.
4. Determine the conductive and convective resistances to heat transfer. Which is dominant in each case?

5. Determine the difference between the centerline thermocouple temperature and the water bath temperature. Make adjustments to correct any difference and explain the reasons for the difference.
6. Explain the trend of the temperature versus time data.
7. Explain what the plots of dimensionless time are showing.
8. Explain the trends or characteristics of the different solids and materials.