Object-free categories

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Abstract

We define categories where the only primitives are arrows (morphisms) and composition. We recover the objects as identity arrows. These results are known [citation needed].

1 Traditional categories and functors

Definition 1 (Traditional Category)**.** A *category* is a tuple $C = (O, A, \mathrm{dom}, \mathrm{cod}, \circ, \infty)$, where

- O and A are classes (*objects* and *arrows*, respectively),
- $\infty \in A$ (the *undefined arrow* or just *undefined*),
- $\mathrm{dom} : (A - \{\infty\}) \to O$ and $\mathrm{cod} : (A - \{\infty\}) \to O$ are functions (*domain* and *codomain*, respectively),
- $\circ : A \times A \to A$ is a binary (infix) operation on A (*composition*),

and the following axioms are satisfied for all $a \in O$ and $f, g, h \in A$:

1. $\infty \circ f = f \circ \infty = \infty$.
2. $(f \circ g) \circ h = f \circ (g \circ h)$ (composition is associative)
3. $f \circ g \neq \infty$ iff $\mathrm{dom}\, f = \mathrm{cod}\, g$.
4. There exists $1_a \in A$ such that $\mathrm{dom}\, 1_a = \mathrm{cod}\, 1_a = a$, and $1_a \circ f = f$ if $\mathrm{cod}\, f = a$ (and $f \neq \infty$), and $g \circ 1_a = g$ if $\mathrm{dom}\, g = a$ (and $g \neq \infty$). (1_a is an *identity on* a.)

Definition 2 (Traditional Functor)**.** Let $C_1 = (O_1, A_1, \mathrm{dom}, \mathrm{cod}, \circ, \infty)$ and $C_2 = (O_2, A_2, \mathrm{dom}, \mathrm{cod}, \circ, \infty)$ be categories. A *functor* from C_1 to C_2 is a tuple $F = (F_o, F_a)$ such that $F_o : O_1 \to O_2$ and $F_a : A_1 \to A_2$ such that the following axioms are satisfied for all $a \in O_1$ and $f, g \in A_1$:

1. $F_a(f) = \infty$ iff $f = \infty$.
2. $F_a(f \circ g) = F_a(f) \circ F_a(g)$.
3. $F_a(1_a) = 1_{F_o(a)}$.

Proposition 1. *If* $f \in A_1 - \{\infty\}$*, then* $F_o(\mathrm{dom}\, f) = \mathrm{dom}\, F_a(f)$ *and* $F_o(\mathrm{cod}\, f) = \mathrm{cod}\, F_a(f)$*.*

Proof. Let $d = \mathrm{dom}\, f$ and let $c = \mathrm{cod}\, f$. We have

$$\infty \neq F_a(f) = F_a(f \circ 1_d) = F_a(f) \circ F_a(1_d) = F_a(f) \circ 1_{F_o(d)}.$$

Thus $\mathrm{dom}\, F_a(f) = \mathrm{cod}\, 1_{F_o(d)} = F_o(d) = F_o(\mathrm{dom}\, f)$. Similarly,

$$\infty \neq F_a(f) = F_a(1_c \circ f) = F_a(1_c) \circ F_a(f) = 1_{F_o(c)} \circ F_a(f).$$

Thus $\mathrm{cod}\, F_a(f) = \mathrm{dom}\, 1_{F_o(c)} = F_o(c) = F_o(\mathrm{cod}\, f)$. □

2 Object-free categories

We will define an *object-free category* to be a tuple $C = (A, \circ, \infty)$, where A is a class (the *arrows*), $\infty \in A$ (*undefined*), and $\circ : A \times A \to A$ is a binary operator (*composition*) on A. We list the axioms of an object-free category together with some definitions.

In the axioms, $f, g, h, \ldots$ are arbitrary elements of A. We say "f is defined" to mean $f \neq \infty$.

Axiom 1. $f \circ \infty = \infty \circ f = \infty$.

Axiom 2. $(f \circ g) \circ h = f \circ (g \circ h)$.

Axiom 3. *If* $f \circ g$ *and* $g \circ h$ *are defined, then so is* $f \circ g \circ h$*.*

Axiom 4. *If* $f_1 \circ g_1$*,* $f_1 \circ g_2$*, and* $f_2 \circ g_1$ *are all defined, then so is* $f_2 \circ g_2$*.*

Definition 3. An arrow $i \in A$ is an *identity* (or *object*) if $i \neq \infty$ and for every $f, g \in A$:

- $i \circ f = f$ if $i \circ f$ is defined, and

- $g \circ i = g$ if $g \circ i$ is defined.

Axiom 5. *If f is defined, then there are identities d and c such that $f \circ d$ and $c \circ f$ are both defined. (And both compositions equal f, of course.)*

We now show how to recover the traditional definition from the object-free definition of a category, identifying objects with their identity maps.

Lemma 1. *For any identities i_1 and i_2, we have $i_1 \circ i_2$ is defined iff $i_1 = i_2$.*

Proof. For the forward direction, if $i_1 \circ i_2$ is defined, then $i_1 = i_1 \circ i_2 = i_2$.

Proving the reverse direction amounts to proving that $i \circ i$ is defined for all identities i. By Axiom 5 with i substituted for f, there is an identity i' such that $i \circ i'$ is defined. But then by the forward direction of the current proposition, $i = i'$, and so $i \circ i$ is defined (and equals i). □

Proposition 2. *If $f \neq \infty$, then the identities d and c of Axiom 5 are uniquely determined by f.*

Proof. Suppose d_1 and d_2 are identities such that $f \circ d_1$ and $f \circ d_2$ are both defined. By Lemma 1, $d_1 \circ d_1$ is defined. Then by Axiom 4, $d_1 \circ d_2$ is defined, whence $d_1 = d_2$ by Lemma 1 again.

A similar proof shows that if c_1 and c_2 are identities and $c_1 \circ f$ and $c_2 \circ f$ are both defined, then $c_1 = c_2$. □

Definition 4. *We define $O \subseteq A$ to be the class of all identities (objects). We define the map $\mathrm{dom} : (A - \{\infty\}) \to O$ to take f to the unique d of Axiom 5. We define the map $\mathrm{cod} : (A - \{\infty\}) \to O$ to take f to the unique c of Axiom 5. We may also write $f : d \to c$.*

Theorem 1. *For all $f, g \in A - \{\infty\}$, $f \circ g$ is defined iff $\mathrm{dom}\, f = \mathrm{cod}\, g$.*

Proof. For the forward direction, let $d = \mathrm{dom}\, f$. Then

$$\infty \neq f \circ g = (f \circ d) \circ g = f \circ (d \circ g),$$

which means that $d \circ g$ must be defined (by Axiom 1). Thus $d = \mathrm{cod}\, g$.

For the reverse direction, suppose $\mathrm{dom}\, f = \mathrm{cod}\, g = i$. Then $f \circ i$ and $i \circ g$ are both defined. Hence by Axiom 3, $f \circ i \circ g$ is defined. But $f \circ i \circ g = f \circ g$. □

Proposition 3. *For any identity i, $\mathrm{dom}\, i = \mathrm{cod}\, i = i$.*

Proof. Immediate from the fact that $i \circ i = i$. □

3 Object-free functors

Definition 5. Let $C_1 = (A_1, \circ, \infty)$ and $C_2 = (A_2, \circ, \infty)$ be object-free categories. An *object-free functor* from C_1 to C_2 is a map $F : A_1 \to A_2$ such that for all $f, g \in A_1$,

1. $F(f) = \infty$ iff $f = \infty$,
2. $F(f \circ g) = F(f) \circ F(g)$, and
3. if f is an identity, then so is $F(f)$.

We can define $F_a := F$ and F_o to be F restricted to O_1, noticing that $F(a) \in O_2$ for all $a \in O_1$. (O_1 and O_2 are the classes of objects in A_1 and A_2, respectively.)

Proposition 4. *For any* $f \in A_1 - \{\infty\}$, $F(\operatorname{dom} f) = \operatorname{dom} F(f)$ *and* $F(\operatorname{cod} f) = \operatorname{cod} F(f)$.

Proof. Let $d = \operatorname{dom} f$ and $c = \operatorname{cod} f$. Then $F(d)$ and $F(c)$ are identities, and

$$\infty \neq F(f) = F(f \circ d) = F(f) \circ F(d),$$
$$\infty \neq F(f) = F(c \circ f) = F(c) \circ F(f).$$

Thus $F(d) = \operatorname{dom} F(f)$ and $F(c) = \operatorname{cod} F(f)$. □