

A Note on a Variant of Immunity, Btt-Reducibility, and Minimal Programs

Preliminary Version

Stephen Fenner*

Department of Computer Science
University of Southern Maine
96 Falmouth Street
Portland, Maine 04103, USA
fenner@cs.usm.maine.edu

Marcus Schaefer[†]

Department of Computer Science
University of Southern Maine
96 Falmouth Street
Portland, Maine 04103, USA
schaefer@cs.uchicago.edu

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Abstract

We define and study a new notion called k -immunity that lies between immunity and hyperimmunity in strength. Our interest in k -immunity is justified by the result that K does not k -tt reduce to a k -immune set which improves a previous result by Kobzev [6]. We apply the result to show that K does not btt-reduce to MIN , the set of minimal programs. Other applications include the set of Kolmogorov random strings, and retraceable and regressive sets. We also give a new characterization of effectively simple sets, and add some results about regressive sets.

Keywords: Computability, bounded reducibilities, minimal programs, immunity.

1 Introduction

There seems to be a large gap between immunity and hyperimmunity (or h -immunity henceforth) that is waiting to be filled. One wonders what happens if the disjoint strong arrays that try to witness that a set is not h -immune are subjected to additional conditions, in particular a condition on the size of the finite sets in the array. Let us first recall the role immunity properties (or thinness properties) have played in computability theory¹.

In 1944 Post published his seminal paper on *Recursively enumerable sets of positive integers and their decision problems* ([6, 8]). In this paper he initiated what has since become known

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¹In this paper we will use the terminology suggested by Soare [9]. In particular we will denote a computably (recursively) enumerable set by c.e. and talk about computable partial functions instead of partial recursive functions.

as Post's program: relating the thinness of a set to its degree, in the hope of showing that there was a c.e. non-computable set with a complement so thin that it could not be Turing complete.

In 1958 Friedberg showed that Post's program had to fail for Turing completeness by constructing a Turing complete maximal, i.e. a set with thinnest possible complement. Notwithstanding this setback Post's idea proved successful for weaker reductions as the following table demonstrates.

If A is	then	Proved in ²
co-immune,	$K \not\leq_m A$.	[Post, 1944]
immune, and B is c.e. and noncomputable,	$B \not\leq_c A$.	
co-immune, B separates a computably inseparable pair,	$B \not\leq_d A$.	Proposition 3.2
k -immune, and B separates a computably inseparable pair,	$B \not\leq_{k-tt} A$.	Theorem 3.4
ω -immune, and B separates a computably inseparable pair,	$B \not\leq_{btt} A$.	Theorem 3.4
ω -immune, and B is c.e. and noncomputable,	$B \not\leq_{bd} A$.	[9]
simple,	$K \not\leq_{btt} A$.	[Post, 1944]
simple, and B is part of a computably inseparable pair,	$B \not\leq_{btt} A$.	[Kobzev, 1973]
hypersimple,	$K \not\leq_{tt} A$.	[Post, 1944]
hypersimple, and B is part of a computably inseparable pair,	$B \not\leq_{tt} A$.	[Denisov, 1974]
hypersimple,	$K \not\leq_{wtt} A$.	[Friedberg, Rogers, 1959]
hyperhypersimple,	$K \not\leq_Q A$.	[Soloviev, 1974], and [Gill, Morris, 1974]
non-computable, semirecursive ³ , η -hyperhypersimple	$K \not\leq_T A$.	[Degtev, 1973], and [Marchenkov, 1976]

In this paper we set out to study the territory between immunity and h-immunity. A set A is called k -immune, if there is no strong disjoint array for which every set in the array intersects A and has cardinality at most k . Then, as the above table indicates $K \not\leq_{k-tt} A$. Calling ω -immune the property of being k -immune for every k , this implies that if A is ω -immune, then $K \not\leq_{btt} A$. In fact we prove a stronger statement which we discovered unaware of the Kobzev result mentioned in the table. Our result, though, gives a more detailed picture. (Kobzev was

²For exact references see [6, 8]

³Note that semirecursiveness is not really a thinness property so this entry does not strictly belong here. There are other properties that imply Turing incompleteness, see for example Harrington and Soare [2].

interested in c.e. sets only, and for those sets k -immunity on the complement is the same as immunity.)

The concepts of k -immunity and ω -immunity have been used implicitly several times in the past, for example by Appel and McLaughlin [1] and Jockusch [3] in their work on retraceable and regressive sets.

In Section 2 we define k -immunity and derive some important basic results. Our main result is presented in Section 3. It implies that K does not k -tt reduce to a k -immune set. We give several applications, notably to MIN , the set of minimal programs (i.e. the minimal indices of a Gödel numbering), showing that K cannot btt-reduce to MIN . In Section 4 we study in some more details the consequences for retraceable and regressive sets. In the last section we have a closer look at ω -immune sets which are not h -immune, and give a new characterization of effectively simple sets.

For notation and definitions of the standard concepts of computability we refer the reader to the usual sources [6, 8].

2 The notion of k -immunity

2.1 Definition

Let $(D_x)_{x \in \omega}$ be a canonical numbering of the finite sets. A family $(D_{f(x)})_{x \in \omega}$ of finite sets is a *disjoint strong array* if f is a computable function, and all the $D_{f(x)}$ are pairwise disjoint and nonempty. It is a *disjoint strong k -array* if it is a disjoint strong array, and in addition $|D_{f(x)}| = k$ for all $x \in \omega$. If we want to emphasize f , we say that f describes a disjoint strong array (or k -array). For shortness we will also sometimes say that the array intersects a set A , which just means that every set in the array intersects A .

We can now define a notion of immunity between immune and h -immune.

Definition 2.1 *A set M is k -immune if there is no disjoint strong k -array, all of whose elements intersect M . That is, there is no computable f such that*

- $(\forall x) |D_{f(x)}| = k,$
- $(\forall x \neq y) D_{f(x)} \cap D_{f(y)} = \emptyset,$ and
- $(\forall x) D_{f(x)} \cap M \neq \emptyset.$

A set M is ω -immune if it is k -immune for all k .

Note that $(k+1)$ -immunity implies k -immunity for all $k \geq 1$, and 1-immunity is the same as immunity. This finer hierarchy of immunity has not attracted a lot of attention. One partial reason is without doubt the following folklore result.

Lemma 2.2 *If \overline{A} is simple, then A is ω -immune.*

Proof. Suppose A is an immune set which is not k -immune for some $k > 1$, and \bar{A} is c.e. We will prove that A is not $(k - 1)$ -immune, which proves the lemma. Let $(D_{f(x)})_{x \in \omega}$ be as in the definition of k -immune. Since A is immune there must be infinitely many x for which $D_{f(x)}$ intersects \bar{A} . Using that \bar{A} is c.e., we can enumerate these infinitely many $D_{f(x)}$ dropping the element which was found to be in \bar{A} . So we can enumerate infinitely many finite sets that are pairwise disjoint, intersect A , and contain at most $k - 1$ elements. In short, A is not $(k - 1)$ -immune. \square

Remark. Jockusch [3] improved the last result considerably, see the paragraph preceding Theorem 4.6.

The notions of k -immunity can be separated in Δ_2^0 though. A much stronger result will be proved in the following section, and some natural examples are presented in Section 2.3.

2.2 Separating the new notions of immunity

In this section we will prove two theorems which show that k -immune sets that are not $(k + 1)$ -immune, and ω -immune sets that are not h -immune are—in a sense—abundant.

Theorem 2.3 *Every c.e., non-computable degree contains a k -immune set which is not $(k + 1)$ -immune (for arbitrary k).*

This theorem is a consequence of a lemma which is stronger, but does not have the dashing good looks of the theorem.

Lemma 2.4 *For every c.e., non-computable set B , and every k there is a k -immune set $A \leq_{\text{wtt}} B$, such that B bounded-disjunctively reduces to A with norm $k + 1$, and A is not $(k + 1)$ -immune.*

Proof. Fix k . We will construct the set A by a finite injury argument. Define a family of intervals $I_e := \{z : e(k + 1) \leq z < (e + 1)(k + 1)\}$. The set A will contain at most one point in every I_e , in fact A will meet the requirement

$$P_e : e \in B \text{ iff } A \cap I_e \neq \emptyset,$$

for all e . This will ensure that B bounded-disjunctively reduces to A with norm $k + 1$. Furthermore if C is an infinite computable subset of B , then the family $(I_e)_{e \in C}$ is a disjoint strong $(k + 1)$ -array which witnesses that A is not $(k + 1)$ -immune. (See also the remark after the proof.)

The set A will furthermore have to meet the requirements

$$N_e : \text{if } (D_{\varphi_e(n)})_{n \in \omega} \text{ is a disjoint strong } k\text{-array,} \\ \text{then there is an } n \text{ such that } D_{\varphi_e(n)} \text{ does not intersect } A,$$

which make A k -immune.

The construction is a finite injury argument with regard to the N type of requirements. Requirement N_e has higher priority than $N_{e'}$ if $e < e'$. Requirements of type P will be fulfilled instantaneously.

During the construction the intervals will be assigned to requirements of type N . At each stage, every interval is assigned to at most one requirement. A requirement N_e has associated with it a *taboo set* T_e of elements it tries to keep out of A . A taboo set contains at most k elements.

Let f be a computable function enumerating B without repetitions.

Stage $s = 0$. Let $A^0 = \emptyset$. Initially all intervals are unassigned, and all the taboo sets T_e are empty.

Stage $2s + 1$. (Satisfy $P_{f(s)}$.) Let $e = f(s)$. We will satisfy P_e . The set $A^{2s} \cap I_e$ is currently empty, since f enumerates without repetition, and only P_e can put an element into I_e . If I_e is currently assigned to a requirement $N_{e'}$, then let a be the smallest element in $I_e - T_{e'}$ (note that this is possible, since I_e contains $k + 1$ elements, whereas $T_{e'}$ contains only k). Should I_e not be assigned to a requirement let a be the smallest element of I_e . Put a into A^{2s+1} . Requirement P_e is satisfied, and will remain so henceforth.

Stage $2s + 2$. (Make A k -immune, and $A \leq_{\text{wit}} B$.) Say N_e *requires attention at stage $2s + 2$* if there are no intervals assigned to N_e , and there exist t and x such that

(i) $\langle e, t, x \rangle < s + 1$, and

(ii) $|D_{\varphi_{e,t}(x)}| = k$, and

(iii) $D_{\varphi_{e,t}(x)}$ does not intersect any interval $I_{e'}$ with $e' < e$, nor does it intersect any interval assigned to a requirement $N_{e'}$ of higher priority, and

(iv) $f(s + 1) < \min(D_{\varphi_{e,t}(x)})$.

Choose the smallest e such that N_e requires attention at stage $2s + 2$. Say requirement N_e *receives attention at stage $2s + 2$* . Let $T_e = D_{\varphi_{e,t}(x)}$. Remove all assignments of intervals to requirements of lower priority than N_e , and assign all intervals that intersect T_e to N_e . For all intervals I_n that are assigned to N_e , and for which $I_n \cap A^{2s+1} \neq \emptyset$, let $I_n \cap A^{2s+2}$ contain only the smallest element of I_n that is not in T_e (again this is possible because of cardinality reasons). For all other intervals I_n let $A^{2s+2} \cap I_n = A^{2s+1} \cap I_n$. Let all $T_{e'} = \emptyset$ for $e' > e$. *End of Construction.*

If we assume that A^s converges to a set A , then we see that by construction all P_e are satisfied: if $e \notin B$, then there will never be an element in $A \cap I_e$; if $e \in B$, then an element is put into the corresponding interval (which is possible, because at any one time there is at most one taboo set associated with the interval, so there is an element of the interval not in the taboo set). During stages $2s + 2$ this element might be moved within the interval, but this does not affect P_e (assuming that A exists).

Every requirement of type N can only be injured by requirements of type N of higher priority, and hence acts only finitely often. Since every interval can only be changed by finitely many requirements this implies that every interval will have a final assignment, or remain

without an assignment. In either case $\lim_{s \rightarrow \infty} A^s \cap I_n$ exists for every n , and hence the A^s converge to a set A for $s \rightarrow \infty$.

Suppose that not all requirements of type N are satisfied. Choose N_e to be the requirement of highest priority that is not satisfied. Then by the argument of the preceding paragraph there is a stage s' after which no requirement $N_{e'}$ of higher priority will act, so N_e will not be injured after stage s' . There are no intervals assigned to N_e at stage s' and $(D_{\varphi_e(x)})_{x \in \omega}$ is a disjoint strong k -array (otherwise N_e would be satisfied, and remain satisfied).

Given y we can effectively find x and t such that conditions (ii) and (iii) of the construction are satisfied, and moreover $\min(D_{\varphi_e(x)}) > y$. If we choose a stage $s > \max\{s', \langle e, x, t \rangle\}$, then all conditions except (iv) are satisfied. Since N_e cannot receive attention (otherwise it would be satisfied permanently), we can conclude that $y < \min(D_{\varphi_e(x)}) \leq f(s'' + 1)$ for all $s'' > s'$. Hence $y \in B$ iff $y \in \{f(0), \dots, f(s')\}$ which is impossible, since B is not computable. So N_e is satisfied.

We are left with the proof that $A \leq_{\text{wtt}} B$. But this follows from the permitting in condition (iv) together with (iii): given y determine the first interval I_n to the right of y (i.e. $y \notin I_n$ and $y \in I_{n-1}$), and use B to determine a stage s such that $\{f(0), \dots, f(s)\} = B \cap \bigcap_{i < n} I_i$. Since f enumerates B without repetitions, from this stage on conditions (iii) and (iv) ensure that A does not change on $\bigcap_{i < n} I_i$, so we can decide A by simulating the construction up to stage s . This gives us a wtt-reduction from A to B . \square

Remark. The above proof showed explicitly that the constructed set A was not $(k + 1)$ -immune. This could have been avoided by use of a lemma from [7], which shows that if a c.e., non-computable set bounded disjointively reduces to a set B with norm $(k + 1)$, then the set B is not $(k + 1)$ -immune.

Using a very similar proof, ω -immunity can be separated from h -immunity. We just state the result without proof. The theorem and the lemma are immediate consequences of Theorem 5.3.

Theorem 2.5 *Every c.e., non-computable degree contains an ω -immune set which is not h -immune.*

Again there is a stronger lemma from which the theorem follows.

Lemma 2.6 *For every c.e., non-computable set B , there is an ω -immune set $A \leq_{\text{wtt}} B$, such that B disjointively reduces to A , and A is not h -immune.*

2.3 Kolmogorov random strings and minimal indices

After showing that our newly defined variant of immunity is non-trivial, it is time we presented some natural examples. The set R of Kolmogorov random strings is defined by $R = \{x : (\forall y \leq x)[\varphi_y(0) \neq x]\}$.

Theorem 2.7 *R is ω -immune but not h -immune.*

We omit the straightforward proof. One might object that R is co-c.e. and hence ω -immune collapses to immunity, so in Section 4 on retraceable and regressive sets we present an easy example of a set in Δ_2^0 which is not co-c.e., ω -immune and not h-immune.

Another natural example, which can be found higher up in the hierarchy, is MIN , the set of minimal indices of a Gödel numbering, i.e. $MIN = \{e : (\forall i < e) \varphi_i \neq \varphi_e\}$. It is well known [5, 7] that MIN is Turing-equivalent to K' , and that it is not h-immune.

Theorem 2.8 (Schaefer [7]) *MIN is ω -immune.*

Proof. The proof generalizes the usual immunity proof for MIN using the k -fold Recursion Theorem. Suppose MIN was not k -immune. Let $D_{f(i)}$ witness this as in the definition of k -immunity. Define a computable function $h(x_1, \dots, x_k) := f((\mu i)[(\forall z \in D_{f(i)})(\forall j)[z > x_j]])$. The function h picks out the index of the first set in the enumeration for which all elements are bigger than any x_j . We use h to define k computable functions. For $1 \leq i \leq k$ let

$$g_i(x_1, \dots, x_k) := \text{the } i\text{th element of } D_{h(x_1, \dots, x_k)}.$$

By the k -fold Recursion Theorem there are k indices e_1, \dots, e_k such that $\varphi_{g_i(e_1, \dots, e_k)} = \varphi_{e_i}$ for all $1 \leq i \leq k$. Since $g_i(e_1, \dots, e_k) > e_i$ this contradicts the fact that $g_i(e_1, \dots, e_k)$ has to be a minimal index for some i . \square

A careful look at the proof shows that MIN is in fact effectively k -immune: there is a total computable function g such that if W_e is a set of canonical indices of pairwise disjoint sets, all of which intersect MIN and contain at most k elements, then $g(e)$ is an upper bound on the cardinality of W_e (in fact it even bounds $\max_{x \in W_e} \{\min(D_x)\}$).

3 Bounded truth-table reducibility

Two sets A and B are *computably inseparable* if there is no computable set C for which $A \subseteq C \subseteq \overline{B}$.

Definition 3.1 *We call a set E a separator if it separates a computably inseparable pair of c.e. sets, i.e. there are c.e. sets A and B which are computably inseparable, and $A \subseteq E \subseteq \overline{B}$.*

It is obvious that a separator cannot be computable. As a consequence of the Low Basis Theorem by Jockusch and Soare [8] we know that there are low separators. Before we turn to bounded truth-table reducibilities we will first show an easy result on disjunctive reducibility that illustrates the power of computably inseparable sets.

Proposition 3.2 *If M is co-immune, and E is a separator, then $E \not\leq_d M$.*

Proof. Let A and B be the pair of c.e. noncomputable sets separated by M , i.e. $A \subseteq M \subseteq \overline{B}$, and f computable such that $x \in E$ iff $D_{f(x)} \cap M \neq \emptyset$. Then $D_{f(x)} \subseteq \overline{M}$ for all $x \in B$, and since \overline{M} is immune, $\bigcup_{x \in B} D_{f(x)}$ is finite. But this contradicts the inseparability of A and B . \square

Since K is half of a computably inseparable pair, the following corollary is immediate.

Corollary 3.3 *If M is simple, then $K \not\leq_d M$.*

Contrast this with the fact that Post's construction of a simple set can be modified so as to yield a conjunctively complete simple set.

We now turn to our real concern in this section, the bounded reducibilities.

Theorem 3.4 *If E is a separator, and M is a k -immune set, then $E \not\leq_{k\text{-tt}} M$.*

This result was proved without knowledge of an earlier, weaker result by Kobzev (see Odifreddi [6, Exercise III.8.10]), which uses very similar techniques.

As above we get the following corollary.

Corollary 3.5 *For all $k \geq 1$ and any set M , if M is k -immune, then $K \not\leq_{k\text{-tt}} M$.*

A stronger statement of this corollary (using the Low Basis Theorem) would have been that there is even a low set which does not k -tt reduce to any k -immune set. We will state all the corollaries for K , but the results also hold for some low set.

In particular, for the sets R and MIN as defined in the last section we have the following corollaries. The first result is already known.

Corollary 3.6 (Kummer [4]) $K \not\leq_{\text{btt}} R$.

Corollary 3.7 $K \not\leq_{\text{btt}} MIN$.

The proof of Theorem 3.4 breaks down into two parts, the general case of k -tt reductions is reduced to a fixed truth table by Lemma 3.9, and the fixed truth table case is resolved in Lemma 3.10.

For the sake of clarity we include the definition of a reduction via a fixed truth table.

Definition 3.8 *Let E and M be arbitrary sets, let $\alpha: 2^k \rightarrow 2$ be a k -ary Boolean function, and let f be computable. We say that E α -reduces to M via f ($f: E \leq_\alpha M$) if*

- $(\forall x) |D_{f(x)}| = k$,
- $(\forall x \in E) \alpha(\chi_M(D_{f(x)})) = 1$, and
- $(\forall x \notin E) \alpha(\chi_M(D_{f(x)})) = 0$,

where $\chi_M(D_y)$ is the vector $(M(y_1), \dots, M(y_k))$, where $D_y = \{y_1 < \dots < y_k\}$. (We say that $E \leq_\alpha M$ if there exists such an f .)

Lemma 3.9 *Suppose E is a separator and M an arbitrary set. If $E \leq_{k\text{-tt}} M$, then there is a separator \tilde{E} and a k -ary Boolean function α such that $\tilde{E} \leq_\alpha M$.*

Proof. Let A and B be two computably inseparable c.e. sets with $A \subseteq E \subseteq \overline{B}$. Since $E \leq_{k\text{-tt}} M$, there is a computable f , and a k -ary Boolean function α_x effectively computable from x such that $|D_{f(x)}| = k$ for all $x \in \omega$, and

- $(\forall x \in A) \alpha_x(\chi_M(D_{f(x)})) = 1$, and
- $(\forall x \in B) \alpha_x(\chi_M(D_{f(x)})) = 0$.

Let $\ell = 2^{2^k}$ and let τ_1, \dots, τ_ℓ enumerate all k -ary Boolean functions. For $1 \leq i \leq \ell$, set $T_i = \{x : \alpha_x = \tau_i\}$. If $A \cap T_i$ and $B \cap T_i$ are computably separable for all i , then clearly A and B are computably separable, so fix i such that $\tilde{A} = A \cap T_i$ and $\tilde{B} = B \cap T_i$ are computably inseparable, and set $\alpha = \tau_i$, and $\tilde{E} = \{x : \alpha(\chi_M(D_{f(x)})) = 1\}$. \square

Lemma 3.10 *If E is a separator and M is k -immune, then there is no k -ary α such that $E \leq_\alpha M$.*

Proof. Let A and B be a computably inseparable pair of c.e. sets with $A \subseteq E \subseteq \overline{B}$. Suppose M is k -immune and there is a k -ary α such that $E \leq_\alpha M$ via some computable f . We can assume without loss of generality that $\alpha(\vec{0}) = 0$, since otherwise we consider $\neg\alpha$ and \overline{E} .

We proceed by *reductio ad absurdum* on k . If $k = 1$, then we must have $\alpha(1) = 1$ by the inseparability of A and B . Thus for all $x \in A$ we have $D_{f(x)} \subseteq M$, and for all $x \in B$ we have $D_{f(x)} \subseteq \overline{M}$. Since each $D_{f(x)}$ is a singleton and M is (1-)immune, $\bigcup_{x \in A} D_{f(x)}$ is finite, which contradicts the inseparability of A and B .

Now suppose $k > 1$, A and B are computably inseparable c.e. sets, M is k -immune, and $f: E \leq_\alpha M$. To complete the proof, we show that there is a separator \tilde{E} such that $\tilde{E} \leq_\beta M$ for some $(k-1)$ -ary β .

Again, assume without loss of generality that $\alpha(0^k) = 0$, so in particular $D_{f(x)} \cap M \neq \emptyset$ for all $x \in A$. There exists a finite set $S \subseteq \omega$ such that $\min(D_{f(x)}) \in S$ for all $x \in A$, since otherwise we could build a disjoint strong k -array of sets intersecting M , contradicting the k -immunity of M . Let $S_0 = S \cap \overline{M}$ and let $S_1 = S \cap M$. For $i \in 2$, define

$$A_i = \{x \in A : \min(D_{f(x)}) \in S_i\},$$

and

$$B_i = \{x \in B : \min(D_{f(x)}) \in S_i\}.$$

Note that all the A_i and B_i are c.e., $A_i \cap B_i = \emptyset$, and the A_i partition A .

Claim 3.11 *There is an $i \in 2$ such that A_i and B_i are computably inseparable.*

Proof of Claim. Suppose not. For $i \in 2$, let C_i be computable with $A_i \subseteq C_i \subseteq \overline{B_i}$. Let

$$C = \bigcup_{i \in 2} (C_i \cap \{x : \min(D_{f(x)}) \in S_i\}).$$

Clearly, C is computable and by definition $A \subseteq C \subseteq \overline{B}$, contradicting the inseparability of A and B . \square **Claim**

Now fixing $i \in 2$ such that A_i and B_i are computably inseparable, we set $\tilde{A} = A_i$, $\tilde{B} = B_i$, and $\beta = \lambda \vec{v} \in 2^{k-1} \cdot \alpha(i, \vec{v})$. Define g such that $D_{g(x)} = D_{f(x)} - \{\min(D_{f(x)})\}$, and finally let $\tilde{E} := \{x : \beta(\chi_M(D_{g(x)})) = 1\}$. Then $\tilde{A} \subseteq \tilde{E} \subseteq \overline{\tilde{B}}$, so \tilde{E} is a separator by the claim, and furthermore $\tilde{E} \leq_\beta M$ via g . \square

Remark. It was observed by Fischer [6] that any btt-reducibility can be transformed into one where the truth table is fixed, i.e., independent of the input. So if one is only interested in btt-reducibility, then we can avoid using Lemma 3.9 and prove that if M is k -immune for all k , then $K \not\leq_{\text{btt}} M$ —a result somewhat weaker than Corollary 3.5, but still strong enough to prove Corollary 3.7.

4 Retraceable and Regressive Sets

Definition 4.1 *A (partial) function g regresses the set A if there is a (not necessarily effective) enumeration $(a_n)_{n \in \omega}$ of A without repetitions such that $g(a_{n+1}) = a_n$ and $g(a_0) = a_0$. If the enumeration can be taken to be in increasing order, then g is said to retrace A . A set is called regressive (retraceable) if it is regressed (retraced) by a computable partial function.*

By the classic result of Dekker and Myhill a retraceable set is either computable or immune, and a regressive set is either c.e. or immune. Regarding regressive sets the following strengthening was shown by Appel and McLaughlin (stated in a different terminology naturally).

Theorem 4.2 (Appel and McLaughlin [1]) *A regressive set is either c.e. or ω -immune.*

Another theorem is immediate from this (although Appel and McLaughlin do not mention this).

Theorem 4.3 *A retraceable set is either computable or ω -immune.*

Proof. A retraceable set is regressive, so it is either c.e. or ω -immune. If it is c.e. it is not immune, and hence computable by the result of Dekker and Myhill. \square

Corollary 4.4 *If A is retraceable, then $K \not\leq_{\text{btt}} A$.*

The theorem also allows us to fulfill the promise of an easy example of a set in Δ_2^0 that is ω -immune without being co-c.e.: let A be the set of initial segments of the characteristic function of K , i.e. $A = \{\sigma \in \{0, 1\}^* : (\forall i < \text{lh}(\sigma))[\sigma(i) = K(i)]\}$, where $\text{lh}(\sigma)$ is the length of the string σ .

Then A is obviously retraced by a total computable function (which deletes the last bit). It is d.c.e., but neither c.e., nor co-c.e., and hence by the preceding theorem ω -immune. It is not h-immune, since it contains a string of every length.

Using a theorem of Kobzev and Lachlan [6] that every btt-complete set is bd-complete we can conclude with the help of theorem 4.2:

Theorem 4.5 *If A is regressive and $K \leq_{\text{btt}} A$, then A is c.e., and hence bd-complete.*

Another notion which simultaneously generalizes enumerability and regressiveness was introduced by Jockusch [3]. A set A is called *uniformly introenumerable* if there is one e such that $\chi_A = \varphi_e^B$ for all infinite subsets B of A . Jockusch proved that every immune uniformly introenumerable set is ω -immune.

Theorem 4.6 *If A is immune and uniformly introenumerable, then $K \not\leq_{\text{btt}} A$.*

The classical results on regressive and retraceable sets roughly speaking show that these sets cannot have easy infinite subsets without being easy themselves. We end this section by a result which argues that the opposite view is also true. Remember that a set A is *effectively immune*, if there is a (total) computable function f such that $W_e \subset A \implies |W_e| < f(e)$ for all e .

Theorem 4.7 *Suppose A has an infinite subset computable in B . If A is regressive and effectively immune, then $A \leq_T B$.*

Proof. Let C be the infinite subset of A with $C \leq_T B$. Suppose A is regressed by the computable partial function g , and strongly effectively immune via f . Let $W_{h(e)} = \{e, g(e), g^2(e), \dots\}$. Define $l(e) = f(h(e))$. To decide whether $e \in A$ search for an element $x \in C$ such that $g^{l(e)}(x) \neq g^{l(e)+1}(x)$ (such x 's are abundant). Then $e \in A$ if and only if there is a k such that $g^k(x) = e$. Namely if $g^k(x) = e$ for some k , then $e \in A$ since $x \in A$ and g regresses A . For the other direction note that if $e \in A$ every regression sequence started on an element of A which is longer than $l(e)$ must run through the sequence $e, g(e), \dots$ \square

A set A is called *introreducible* if it is computable in all its infinite subsets, and *uniformly introreducible*, if there is one oracle algorithm that computes A with any infinite subset of A as an oracle. Note that the above proof was uniform in C .

Corollary 4.8 *If a set is regressive and effectively immune, then it is uniformly introreducible.*

We note that in Theorem 4.7 the condition that A be strongly effectively immune could have been dropped, to still get the conclusion that $A \leq_T B \oplus K$, since in the proof we can use the K oracle to compute an upper bound on $|W_{h(e)}|$.

Corollary 4.9 *Suppose A has an infinite subset computable in B . If A is regressive, then $A \leq_T B \oplus K$.*

This can be put more succinctly by letting A be Σ_k complete and $B = \emptyset^{(k-1)}$ for $k \geq 2$.

Corollary 4.10 *No Σ_k -complete set is regressive for $k \geq 2$.*

We can apply the corollary to MIN, since it is Σ_2 -complete.

Corollary 4.11 *MIN is not regressive.*

5 Fat and thin sets

We already mentioned earlier (without proving it) that there are ω -immune sets which are not h -immune. There is still a very large gap between hyperimmunity and ω -immunity which we will investigate in this section. We first need some notation. Let $(D_{f(x)})_{x \in \omega}$ be a disjoint strong array. Then define $(\#f)(n) = |\{x : |D_{f(x)}| = n\}|$, i.e. $\#f$ counts how many sets of cardinality n appear in the array. Since we will restrict our attention to disjoint strong arrays intersecting ω -immune sets, $\#f$ only takes finite values. Remember that g *dominates* f , if $g(n) > f(n)$ for almost all n . If the inequality holds only infinitely often, then g is said to *majorize* f .

Definition 5.1 An ω -immune set A is called thin, if there is a computable function which dominates every $\#f$, where $(D_{f(x)})_{x \in \omega}$ is a disjoint strong array intersecting A that is $A \cap D_{f(x)} \neq \emptyset$ for all x .

The intuition is that although A is not h-immune, it is so thin that we can effectively bound (eventually) the number of sets of a given cardinality in any disjoint strong array intersecting A . The other extreme is a set which allows for disjoint strong arrays beating any computable bound.

Definition 5.2 An ω -immune set A is called fat, if there is a disjoint strong array $(D_{f(x)})_{x \in \omega}$ intersecting A such that $\#f$ dominates every computable function.

Playing with quantifiers gives several intermediate degrees of fatness and thinness. We will only show that the two notions defined above are proper. Obviously no set can be fat and thin at the same time.

5.1 The fat sets

In this section we will prove the existence of fat sets in every c.e. noncomputable degree. So far we have not been able to find natural examples of fat sets.

Theorem 5.3 Every c.e. noncomputable degree contains a fat set.

As always the theorem follows from a lemma.

Lemma 5.4 For every c.e., non-computable set B , there is a fat set $A \leq_{\text{wtt}} B$, such that B disjunctively reduces to A .

Note that this implies Theorem 2.5 and Lemma 2.6 since fat sets are ω -immune but not h-immune.

Proof. By a finite injury argument we will construct a set A , and a computable function f such that $(D_{f(x)})_{x \in \omega}$ is a strong disjoint array intersecting A . There are three types of requirements. The P requirements code B into A on the even elements, while the R and N requirements guarantee that A is fat if restricted to the odd elements (hence A is fat). The negative N requirements make A ω -immune.

Let $F_e = \{2^{e+1}, 2^{e+1} + 2, \dots, 2^{e+2} - 2\}$. Note that $|F_e| = 2^e$ and F_e contains only even numbers. We fulfill

$$P_e : e \in B \text{ iff } A \cap F_e \neq \emptyset,$$

for all e . This will ensure that B disjunctively reduces to A .

To assure that A intersects enough sets in the strong disjoint array we satisfy

$$\begin{aligned} R_{e,k} : & \text{ if } \varphi_e(k) \downarrow \text{ and } k \geq e, \text{ then} \\ & (\#f)(k) \geq \varphi_e(k). \end{aligned}$$

Finally we have to make sure that A is ω -immune.

N_e : if $(D_{\varphi_e(n)})_{n \in \omega}$ is a disjoint strong e -array,
then there is an n such that $D_{\varphi_e(n)}$ does not intersect A ,

The construction is a finite injury argument with regard to the N type of requirements. The priority ordering is

$$\begin{array}{ccccccc}
& R_{0,0} & & & & P_0 & < N_0 \\
< & R_{0,1} & < R_{1,1} & & & P_1 & < N_1 \\
& & & & & \vdots & \\
< & R_{0,k} & < \cdots & < R_{k,k} & < P_k & < N_k
\end{array}$$

Requirements of type P and R will be fulfilled instantaneously and never be injured afterwards.

During the construction collections of finite sets of integers will be assigned to requirements of all three types. These sets will be pairwise disjoint. More exactly F_e (containing even integers) will be assigned to P_e when e enters B . Both R and N type requirements will be assigned more than one set in general. Sets assigned to $R_{e,k}$ will contain k elements. Sets assigned to N_e will contains more than e elements. Furthermore a requirement N_e has associated with it a *taboo set* T_e of at most e elements it tries to keep out of A .

Let g be a computable function enumerating B without repetitions.

Stage $s = 0$. Let $A^0 = \emptyset$, f undefined everywhere, $u^0 = 1$. Initially nothing is assigned to any R_e , and all the taboo sets T_e are empty.

Stage $3s + 1$. (Satisfy $P_{g(s)}$.) Let $e = g(s)$. We will satisfy P_e . The set $A^{3s} \cap F_e$ is currently empty, since g enumerates without repetition, and only P_e can put an element of F_e into A . Consider the set $T = \bigcup_{i=0}^{e-1} T_i$. Then T contains at most $(e-1)e/2 < 2^e = |F_e|$ elements, so we can choose the smallest $a \in F_e - T$ and enumerate a into A , i.e. $A^{3s+1} \cap F_e = \{a\}$, and $A^{3s+1} - F_e = A^{3s}$. Assign F_e to P_e . Let $u^{3s+1} = u^{3s}$. Requirement P_e is satisfied, and will remain so henceforth. Reset all taboo sets of N requirements of lower priority, i.e. $T_i = \emptyset$ for all $i \geq e$.

Stage $3s + 2$. (Satisfy R type requirements.) Say that $R_{e,k}$ *requires attention at stage* $3s + 2$ if $\varphi_{e,s}(k) \downarrow$ and $k \geq e$ and $R_{e,k}$ has not received attention before. Let $R_{e,k}$ be the highest priority requirement requiring attention. We say that $R_{e,k}$ *receives attention*. Let $\varphi_{e,k} = y$, and $H_i = \{u^{3s+1} + i * 2k, u^{3s+1} + 2 + i * 2k, \dots, u^{3s+1} + (i + 1) * 2k - 2\}$ for $0 \leq i < y$ that is the H_i split up the first $y * k$ odd integers beyond u^{3s+1} into y blocks of length k . Assign $(H_i)_{0 \leq i < y}$ to requirement $R_{e,k}$. Enumerate the smallest element of each H_i into A^{3s+2} , i.e. $A^{3s+2} \cap H_i = \{u^{3s+1} + i * 2k\}$, and leave the rest of A unchanged. Extend f to include canonical indices of all the H_i ($0 \leq i < y$) in its range. Let $u^{3s+2} = u^{3s+1} + y * 2k$. Requirement $R_{e,k}$ is satisfied, and will remain so henceforth. Reset all taboo sets of N type requirements of lower priority, i.e. $T_i = \emptyset$ for all $i \geq k$.

Stage $3s + 3$. (Satisfy N type requirements, and $A \leq_{\text{wtt}} B$.) Say N_e requires attention at stage $3s + 3$ if there is no collection of sets assigned to N_e , and there exist t and x such that

- (i) $\langle e, t, x \rangle < s + 1$, and
- (ii) $|D_{\varphi_{e,t}(x)}| \leq e$, and
- (iii) $D_{\varphi_{e,t}(x)}$ lies completely to the right of any set assigned to any requirement (P , R or N) of higher priority, i.e. if $z \in D_{\varphi_{e,t}(x)}$ and $w \in H$ where H is a set assigned to a requirement of higher priority, then $z > w$, and
- (iv) $g(s + 1) < \min(D_{\varphi_{e,t}(x)})$.

Choose the smallest e such that N_e requires attention at stage $3s + 3$. Say requirement N_e receives attention at stage $3s + 3$. Let $T_e = D_{\varphi_{e,t}(x)}$. Remove all assignments of collections of sets to N type requirements of lower priority than N_e , and assign all sets assigned to R or P type requirements that intersect T_e to N_e . Note that condition (iii) implies that these sets belong to R type requirements of lower priority than N_e , so they contain more than e elements. For every set H that is assigned to N_e , and for which $H \cap A^{3s+3} \neq \emptyset$, let $H \cap A^{3s+3}$ contain only the smallest element of H that is not in T_e (this is possible, since $|H| > e$ and $|T_e| \leq e$). Otherwise A^{3s+3} remains unchanged. Let all $T_{e'} = \emptyset$ for $e' > e$, and $u^{3s+3} = 1 + 2 \max\{u^{3s+2}, \max(T_e)\}$.
End of Construction.

If we assume that A^s converges to a set A , then we see that by construction all P_e are satisfied: if $e \notin B$, then there will never be an element in $A \cap F_e$; if $e \in B$, then an element from F_e is put into A without injuring any higher priority N type requirements as argued during the construction. During stages $3s + 3$ this element might be moved within F_e , but this does not affect P_e (assuming that A exists).

Likewise we argue that every $R_{e,k}$ will be satisfied if A^s converges to a set A . (Note that since u^s takes on only odd values the P and R type requirements do not interfere with each other.)

Let H be a set assigned to a P or R type requirement. We have to argue that $H \cap A^s$ converges. Since g enumerates B without repetition $\liminf_{s \rightarrow \infty} g(s) = \infty$, so there is a stage after which no taboo set can intersect H (by condition (iv)). So $H \cap A^s$ will remain unchanged from this stage onwards. Since only elements from sets assigned to R or P type requirements are ever enumerated into A this proves that $\lim_{s \rightarrow \infty} A^s = A$ exists.

So all P and R requirements will be fulfilled. Furthermore note that every P and R requirement acts at most once.

Suppose that not all requirements of type N are satisfied. Choose the N_e of highest priority which is not satisfied. Requirements of type N can be injured by all other types of requirements. Since P and R requirements act at most once it can be proved by induction that every N type requirement is injured at most finitely often, so there is a stage s' after which no P or R type requirement of higher priority than N_e will act, and no N_i with $i < e$ will require attention.

In particular the sets assigned to requirements of higher priority than N_e will not change after stage s' .

There are no sets assigned to N_e at stage s' and $(D_{\varphi_e(x)})_{x \in \omega}$ is a disjoint strong e -array (otherwise N_e would be satisfied, and remain satisfied).

Given y we can effectively find x and t such that conditions (ii) and (iii) of the construction are satisfied, and moreover $\min(D_{\varphi_e(x)}) > y$. If we choose a stage $s > \max\{s', \langle e, x, t \rangle\}$, then all conditions except (iv) are satisfied. Since N_e cannot receive attention (otherwise it would be satisfied permanently), we can conclude that $y < \min(D_{\varphi_e(x)}) \leq g(s'' + 1)$ for all $s'' > s'$. Hence $y \in B$ iff $y \in \{g(0), \dots, g(s')\}$ which is impossible, since B is not computable. So N_e is satisfied.

We are left with the proof that $A \leq_{\text{wtt}} B$. But this follows from the permitting in condition (iv) together with (iii): given y simulate the construction of A until in some stage $3s + 2$ a set H which lies completely to the right of y is assigned to a requirement $R_{e,k}$, i.e. if h is the smallest element of H , then $y < h$. Use B to determine a stage $t > 3s + 2$ such that $\{g(0), \dots, g(t)\} = B \setminus h$. Since g enumerates B without repetitions, from this stage on conditions (iii) and (iv) ensure that A does not change below h , so we can decide whether $y \in A$ by continuing to simulate the construction of A up to stage t . This gives us a wtt-reduction from A to B . \square

5.2 The thin sets

At the other end of the spectrum we have thin sets. Thin sets occur more naturally than fat sets, and instead of another tedious degree construction we give two very different examples of thin sets. Both examples will fulfill a stronger property than thinness: rather than one computable function that dominates every counting function eventually, there will be an algorithm that finds an (everywhere) dominating function uniformly for every strong disjoint array. To make this precise:

Definition 5.5 *A set A is uniformly thin if there is a total computable function f such that $(\#\varphi_e)(n) \leq f(e, n)$ for every strong disjoint array $(D_{\varphi_e(x)})_{x \in \omega}$.*

Lemma 5.6 *Every uniformly thin set is thin.*

Proof. Suppose A is uniformly thin via f . Let $d(n) = \max\{f(e, n) : e \leq n\}$. If φ_e describes a disjoint strong array, then $(\#\varphi_e)(n) \leq d(n)$ for every $n \geq e$. \square

Sets which are c.e. and have a uniformly thin complement turn out to be good old friends.

Definition 5.7 *A set A is effectively simple, if it is c.e. and there is a computable function f such that*

$$W_e \subset \bar{A} \implies |W_e| < f(e).$$

Theorem 5.8 *The complement of an effectively simple set is (uniformly) thin.*

Before proving the theorem we note several consequences. First of all the theorem gives us a host of natural examples of (uniformly) thin sets like the complement of Post's simple set, and R . Secondly it yields an alternative characterization of effectively simple sets.

Corollary 5.9 *A c.e. set is effectively simple if and only if its complement is uniformly thin.*

One direction is covered by the theorem, the other is trivial. The corollary might be helpful in separating the notions of uniformly thin and thin. All that would be necessary is to construct a simple set which is not effectively simple, and whose complement is thin.

The theorem is a strong effective version of the folklore result (Lemma 2.2) we presented in the beginning. The proof presented there can be recast to show that it really depends on the non-uniform knowledge of two parameters $m = \limsup_{x \rightarrow \infty} |D_{g(x)} \cap \bar{A}|$ and the point n such that $|D_{g(x)} \cap \bar{A}| \leq m$ for $x \geq n$. These are serious obstacles for an effectivization, and it takes some work to overcome them.

Proof of Theorem 5.8. Let \bar{A} be effectively simple, hence \bar{A} is c.e. and there is a computable function f such that $W_e \subset A \implies |W_e| < f(e)$. We write \bar{A}_s for an effective approximation to \bar{A} .

Call g a *partial disjoint strong n -array* for A , if

- (i) $D_{g(x)} \cap D_{g(y)} = \emptyset$ for all $x, y \in \text{dom}(g)$, and
- (ii) $|D_{g(x)}| = n$ for all $x \in \text{dom}(g)$,
- (iii) $D_{g(x)} \cap A \neq \emptyset$ for all $x \in \text{dom}(g)$.

We claim that there is a computable (total) function h such that $|\text{dom}(g)| < h(e, n)$ for every $g = \varphi_e$ which is a partial disjoint strong n -array for A . With this claim we can finish the proof of the theorem. Define a computable function $k(e, n)$ by

$$\varphi_{k(e, n)}(x) = \begin{cases} g(x) & \text{if } |D_{\varphi_e(x)}| = n, \\ \uparrow & \text{else,} \end{cases}$$

and let $f(e, n) = h(k(e, n), n)$. Suppose $g = \varphi_e$ describes a strong disjoint array intersecting A . Then $(\#g)(n) = |\text{dom}(k(e, n))| < h(k(e, n), n) = f(e, n)$, so f witnesses that A is uniformly thin, hence it is thin.

For the proof of the claim assume that $g = \varphi_e$ is a partial disjoint strong n -array for A . (In the following note that even if φ_e is not, the algorithm for $h(e, n)$ will compute some (meaningless) value.)

To compute $h(e, n)$ we will use an intermediate function $h'(e, n, m)$ defined by induction on m . The idea is that $h'(e, n, m)$ computes a correct upper bound on $|\text{dom}(g)|$ if we know that $\max\{|D_{g(x)} \cap \bar{A}| : x \in \text{dom}(g)\} \leq m$.

It is easy to compute $h'(e, n, 0)$ because we can assume that $D_{g(x)} \subset A$ for all $x \in \text{dom}(g)$. So using f we can effectively determine an upper bound on $|\text{dom}(g)|$. Let us show how to compute $h'(e, n, m)$ inductively from $h'(\cdot, n, m - 1)$.

We know that $\max\{|D_{g(x)} \cap \overline{A}| : x \in \text{dom}(g)\} \leq m$. Consider the set $C = \{y_{m+1}, \dots, y_n : (\exists x)[D_{g(x)} = \{y_1, \dots, y_n\}, y_1, \dots, y_m \in \overline{A}]\}$. Then C is a c.e. subset of A , and we can find an index for it (effectively in n and uniformly in g). So using f we get an upper bound c on the cardinality of C .

We compute a (partial) function $r(i)$ as follows: search for the smallest $\langle x_1, \dots, x_i, s \rangle$ for which x_1, \dots, x_i are pairwise different, and $D_{\varphi_{e,s}(x_j)} \cap \overline{A}_s$ contains at least m elements for every $1 \leq j \leq i$. Let $r(i) = \max\{g(x_j) : 1 \leq j \leq i\} + 1$.

The computation of $r(|C|)$ converges and if $x > r(|C|)$, and $x \in \text{dom}(g)$, then $D_{g(x)}$ contains at most $m - 1$ elements from \overline{A} . If we knew $|C|$ we could compute h' recursively. Unfortunately we do not know the correct value of $|C|$, so we have to consider the c functions ($0 \leq i < c$):

$$\varphi_{e(i)}(x) = \begin{cases} g(x + r(i)) & \text{if } r(i) \text{ is defined,} \\ \uparrow & \text{else,} \end{cases}$$

Every such $\varphi_{e(i)}$ describes a partial disjoint strong n -array for A , and one of them fulfills the inductive condition. So we will get an upper bound if we consider all possibilities, i.e. let $h'(e, n, m) = \max_{0 \leq i < c} \{r(i) + h'(e(i), n, m - 1)\}$. Finally defining $h(e, n) = h'(e, n, n - 1)$ finishes the proof of the claim. \square

There is one more natural example for a thin set we should mention.

Theorem 5.10 *MIN is (uniformly) thin.*

We leave the proof to the reader. The basic observation is that MIN is effectively k -immune uniformly in k .

6 Conclusion

We have shown that k -immunity is a powerful tool in the study of bounded-truth table reductions and establishes some hitherto unknown connections (between retraceability and btt-hardness for example). We believe that by defining k -immunity explicitly we have isolated a helpful notion that has been used implicitly in results of the past. Further research could aim at collecting and unifying some of these results. It seems, for example, that k -immunity might play a role in the study of effective versions of Ramsey's Theorem or in Frequency computation.

We also provided some initial results on ω -immune sets which are not hyperimmune, and showed that there is a variety of sets residing in this realm, in particular the thin sets which generalize effectively simple sets.

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