Monday, September 5, 2022 2:52 PM

CSCE 750
9/13/22 (lecture 2)

DSB(A[1..n])
★ k = random integer in {1,...,lg n}
for i = 1 to k do
  j = random(1,...,n) ←
  A[j] = DSS(A[j], n) ← Takes $\Theta(n)$ time
end for
return A

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For given k, k iterations of the for-loop
Each iteration takes $\Theta(n)$ time (dominated by DSS call)
Total runtime is $\Theta(kn)$
Each k value has prob $\frac{1}{\lg n} = Pr[k]$

Let E(n) be the expected runtime of DSB on an array of size n.

$\boxed{\text{Recall}}$: $E(X) = \sum_x x \cdot Pr\boxed{X=x}$

$$E(n) = \sum_{k=1}^{\lg n} kn\left(\frac{1}{\lg n}\right)$$
$$= \frac{n}{\lg n}\underbrace{\sum_{k=1}^{\lg n} k}$$
$$= \frac{n}{\cancel{\lg n}} \frac{\cancel{\lg n}(\lg n + 1)}{2}$$
$$= \Theta(n \lg n)$$

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Worst-case expected time of Randomized Quicksort

E(n) is E(T(n))

$$E(n) = \underline{\Theta(n)} + \underbrace{\frac{1}{n}}_{\text{prob of any particular q-value}} \sum_{q=0}^{n-1}\left(E(q) + E(n-q-1)\right)$$

$$\boxed{E(n) = \Theta(n) + \frac{2}{n}\sum_{q=0}^{n-1} E(q)}$$

Guess that $E(n) = O(n \lg n)$
Pf by subst method

$E(m) \le c\, m \ln m$ for $m < n$

$$E(n) \le an + \frac{2}{n}\sum_{q=0}^{n-1} E(q)$$
$$\le an + \frac{2c}{n}\sum_{q=0}^{n-1} q \ln q$$
$$\le an + \frac{2c}{n}\int_1^n x \ln x \, dx$$
$$= an + \frac{2c}{n}\left[\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right]_1^n$$
$$= an + \frac{\cancel{2}c}{\cancel{n}}\left(\frac{n^{\cancel{2}} \ln n}{\cancel{2}} - \frac{n^2}{4} + \frac{1}{4}\right)$$
$$= an + cn \ln n - c\left(\frac{n^2 - 1}{2n}\right)$$
$$\le cn \ln n \quad [c > 3a]$$

$\therefore E(n) = O(n \ln n) = O(n \lg n)$