Tuesday, September 6, 2022 2:37 PM

CSCE 750
9/6/2022

Master Theorem

Recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $a > 0$ constant
$b > 1$ constant
and $f(n)$ is asymptocally positive.

Master theorem

small f(n)
1. If $f(n) = O(n^t)$ for some constant $t < \underbrace{\log_b a}$, then $T(n) = \Theta(n^{\log_b a})$

med. f(n)
2. If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \lg n)$

big $f(n)$

3. If $f(n) = \Omega(n^t)$ for some $t > \log_b a$, [and $\exists c < 1$, $a f(\frac{n}{b}) \le c f(n)$ for all sufficiently large n], then

$$T(n) = \Theta(f(n)).$$

Ex: $T(n) = 2T(\frac{n}{2}) + n^1$

(mergesort recurrence)

- First thing: find $\log_b a$

$$\left.\begin{array}{l} a=2 \\ b=2 \end{array}\right] \log_b a = \lg 2 = 1$$

- Next: decide which case

$$f(n) = n = n^1 = n^{\log_b a} = \Theta(n^{\log_b a})$$

$\therefore$ case 2 (medium f)

$\therefore$ $T(n) = \Theta(n \lg n)$

$$T(n) = 9T(\frac{n}{3}) + n$$

$$\left.\begin{array}{l} a=9 \\ b=3 \end{array}\right\} \log_b a = \log_3 9 = 2$$

$f(n) = n = n^1 \quad (1 < 2)$

$= O(n^1)$

$\therefore$ case 1 (small f)

$T(n) = \Theta(n^2)$

$T(n) = T\left(\frac{2n}{3}\right) + 1$

$a = 1$

$b = 3/2$

$$\frac{2n}{3} = \frac{n}{b}$$

$$= \frac{n}{3/2}$$

$\log_{3/2} 1 = 0$

$f(n) = 1 = \Theta(n^0)$

$= \Theta(n^{\log_b a})$

$\therefore$ case 2

$T(n) = \Theta(\lg n)$

$T(n) = 3T\left(\frac{n}{4}\right) + n \lg n$

$$\left.\begin{array}{l} a = 3 \\ b = 4 \end{array}\right] \log_b a = \log_4 3$$

$< \log_4 4 = 1$

$f(n) = n \lg n = \Omega(n)$

$= \Omega(n^1)$

$\therefore$ case 3 (big f)

$T(n) = \Theta(f(n)) = \Theta(n \lg n)$

→ $T(n) = 2T(\frac{n}{2}) + n \lg n$

$a = b = 2$, $\log_b a = \log_2 2 = 1$

$f(n) = n \lg n = n^1 \lg n$

$n \lg n = \Omega(n)$ so can't be case 1

$n \lg n = \omega(n)$, so $f(n)$ is too big for case 2

for case 3, must have $f(n) = \Omega(n^t)$ for some $t > \log_b a = 1$

compare

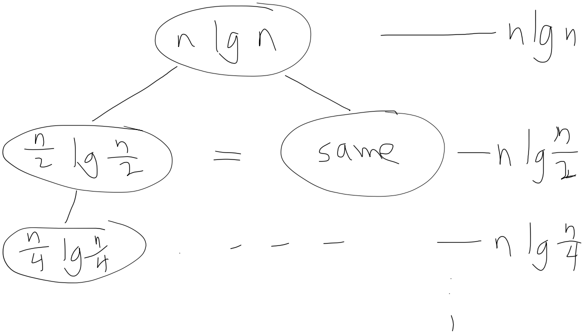
$\frac{n \lg n}{n}$ with $\frac{n^t}{n}$ $(t>1)$

$$\lg n = o(n^{t-1}) \quad (t-1>0)$$

not Ω

not case 3

$\therefore$ Master thm does not apply.



level i ——— $n\lg\frac{n}{2^i}$

Balanced tree with depth lg n

$$T(n) = \sum_{i=0}^{\lg n} n\lg\left(\frac{n}{2^i}\right) = n\sum_{i=0}^{\lg n}(\lg n - \underbrace{\lg 2^i}_{i})$$

$$= n\sum_{i=0}^{\lg n}(\lg n - i)$$

$$= n\left(\sum_{i=0}^{\lg n}\lg n - \sum_{i=0}^{\lg n} i\right)$$

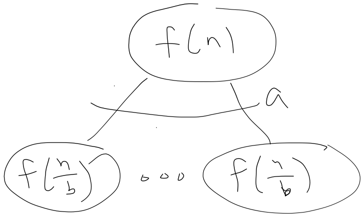
$= \ldots$

"Proof" of the Master Thm using the tree method:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

Set $s := \log_b a$

Case 1: $f(n) = O(n^t)$ for some constant $t < s$,



$\le cn^t$

$\le a\left(\frac{n}{b}\right)^t$

$\le a^2\left(\frac{n}{b^2}\right)^t$

level i ——— $\le a^i\left(\frac{n}{b^i}\right)^t$

$\log_b n$ many levels

$$T(n) \le \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^t}\right)^i n^t = n^t \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{\underline{b^t}}\right)^i$$

>1

[case 1 ays $t < s = \log_b a$, so

so his means

$$b^t < b^s = \cancel{b^{\log_b}} a = a$$

so $\frac{a}{b^t} > 1$, so

$$T(n) \quad n^t \left(\frac{\left(\frac{a}{b^t}\right)^{\log_b n} - 1}{\left(\frac{a}{b^t}\right) - 1} \right)$$

$$= \Theta\left(n^t \left(\left(\frac{a}{b^t}\right)^{\log_b n} - 1 \right) \right)$$

$$= \Theta\left(n^t \left(\frac{a}{b^t}\right)^{\log_b n} \right)$$

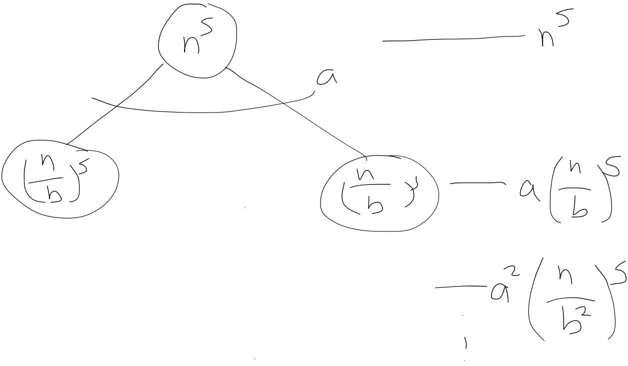
$$= \Theta\left(n^t \frac{a^{\log_b n}}{\left(\cancel{b^{\log_b n}}\right)^t} \right)$$

$$= \Theta\left(a^{\log_b n} \right)$$

$$= \Theta\left(n^{\log_b a} \right) = \Theta\left(n^s \right)$$

Need to fix lower bound

Case 2 : $f(n) = \Theta(n^5)$


n^5
a
$\left(\frac{n}{b}\right)^5$
$\left(\frac{n}{b}\right)^5$
n^5
$a\left(\frac{n}{b}\right)^5$
$a^2\left(\frac{n}{b^2}\right)^5$

$$T(n) = \Theta\left(\sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n^5}{(b^5)^i}\right)\right)$$

$$= \Theta\left(\sum_{i=0}^{\log_b n - 1} \underbrace{\left(\frac{a}{b^5}\right)^i}_{=1} n^5\right)$$

$$\boxed{b^5 = b^{\log_b a} = a}$$

$$= \Theta\left(\sum_i n^5\right)$$

$$= \Theta(n^5 \log_b n)$$

$$= \Theta(n^5 \lg n) \quad //$$

case 3 : $f(n) \geq n^t \quad (t > s)$

$$T(n) \geq \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^t}\right)^i n^t$$

$$= n^t \underbrace{\sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^t}\right)^i}_{}$$

Will fix next time