CSCE 750
9/21/2023

Mixed arithmetic/geometric sums

$$S = \sum_{j=0}^{n-1} j \cdot r^j = \sum_{j=1}^{n-1} j r^j \qquad (r \neq 1)$$

$$rS = \sum_{j=0}^{n-1} j \cdot r^{j+1} = \sum_{j'=1}^{n} (j'-1) r^{j'} = \sum_{j=1}^{n} (j-1) r^j$$

$$= \sum_{j=1}^{n} j r^j - r \sum_{j=0}^{n-1} r^j = \sum_{j=1}^{n} j r^j - r\left(\frac{1-r^n}{1-r}\right)$$

$(1-r)S$

$\parallel$

$$S - rS = r\left(\frac{1-r^n}{1-r}\right) - nr^n$$

$$S = r\left(\frac{1-r^n}{(1-r)^2}\right) - \frac{nr^n}{1-r}$$

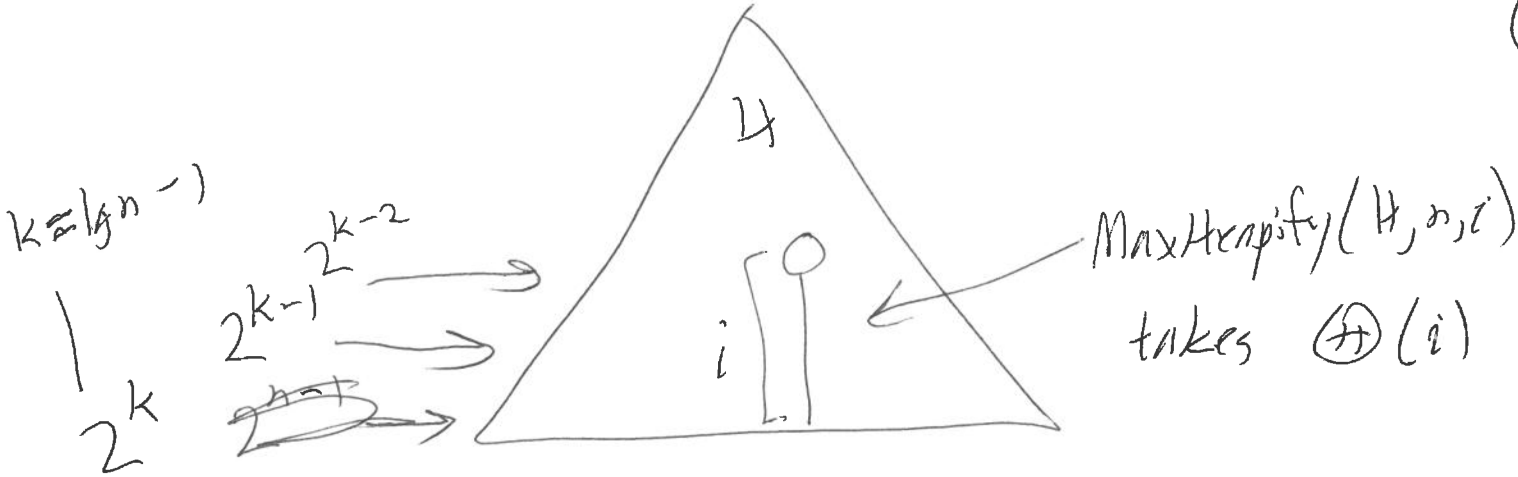
(Assume $|r|<1$)

$$\lim_{n\to\infty} S = \sum_{j=1}^{\infty} j r^j = \lim_{n\to\infty} r\left(\frac{1-\cancel{r^n}}{(1-r)^2}\right) - \frac{\cancel{nr^n}}{1-r}$$

$r^n \to 0$, $nr^n \to 0$

$$= \frac{r}{(1-r)^2} < \infty$$

Analysis of ~~H~~ BuildMaxHeap(H):



$$\text{Total time} = \sum_{i=0}^{\lg n - 1} \underbrace{\Theta(i)}_{ci} \cdot \underbrace{(\#\text{ nodes at height } i)}_{2^{k-i}}$$

$$= c2^k \sum_{i=0}^{\lg n - 1} i 2^{-i} \leq c2^k \sum_{i=0}^{\infty} i 2^{-i}$$

$$(r = \tfrac{1}{2}) \qquad = c2^k \left(\frac{1/2}{(1-\frac{1}{2})^2} \right) = 2c2^k$$

$$= 2c2^{\lg n} = 2cn$$

$$= \Theta(n)$$

$\boxed{\text{Time} = \Omega(n) \text{ because MaxHeapify is called } \approx \tfrac{n}{2} \text{ times}}$

Bernoulli trials
Binomial Dist.
Geometric Distrib.

A Bernoulli trial is any random experiment with only 2 outcomes (e.g., H/T (heads or tails), success/failure, ...)

Let $p := \Pr[H]$

$q := 1 - p = \Pr[T] \qquad (0 \leq p \leq 1)$

A ~~bin~~ binomial experiment is a sequence of n identical Bernoulli trials, each independent of the others.

Binomial Distribution

$$b(k; n, p) = \Pr[k \text{ many heads } \& (n-k) \text{ tails}]$$

$$p = \Pr[\text{heads}]$$

$$q = \Pr[\text{tails}]$$

1 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ n

H H T H H H T T - - - - T

p p q p p p q q $\qquad\qquad\qquad$ q

Pr[any given sequence of k heads and n-k tails]

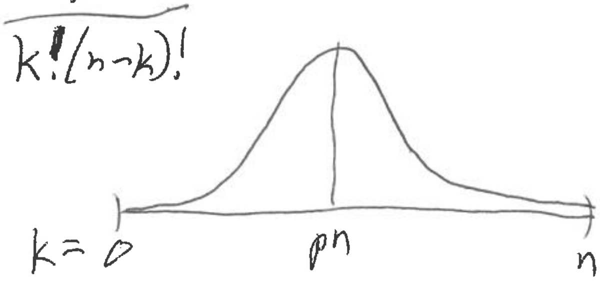
is $\boxed{p^k q^{n-k}}$

$$\Pr[k \text{ exactly heads}] = p^k q^{n-k} (\# \text{ of seq. w/ } k \text{ heads})$$

$$= \boxed{\binom{n}{k} p^k q^{n-k}} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

"n choose k"

$$E[\#\text{heads}] = pn$$



Geometric Distribution

Run identical, indep. Bernoulli trials until see first heads,

Geom. dist is # trials to get 1st heads

$(k>0)$ $\Pr\{k \text{ trials}\} = q^{k-1} p$

$$E[\#\text{ of trials}] = \sum_{k=1}^{\infty} k \cdot \Pr[k \text{ trials}]$$

$$= \sum_{k=1}^{\infty} k q^{k-1} p = \frac{p}{q} \underline{\sum_{k=1}^{\infty} k q^{k}} = \frac{p}{q}\left(\frac{q}{(1-q)^2}\right)$$

$$= \frac{p}{q}\left(\frac{\cancel{q}}{p^2}\right) = \boxed{\frac{1}{p}}$$