CSCE 750
9/19/2023

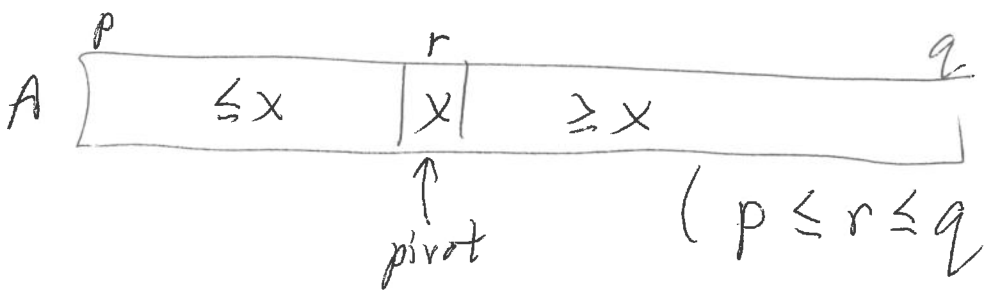
Randomized algos & their analysis

Quicksort$(A[p..q])$ $\quad n = q-p+1$

$T(n)$ if $p < q$:

$O(1)$ — Choose a ~~pivot~~ pivot value from $A[p..q]$, say, $A[m]=x$ $(p \le m \le q)$

$\Theta(n)$ — Partition so that



$(p \le r \le q)$

$T(r-p)$ — Quicksort$(A[p..(r-1)])$

$T(q-r)$ — Quicksort$(A[(r+1)..q])$

?? How to choose the pivot?

Worst-case analysis of quicksort, when $r=p$ or $r=q$,

say, $r=p$

$T(n) = \Theta(n) + T(n-1)$

$T(n) = \Theta(n^2)$

Average-case ~~any~~ analysis:

Assume, given input size n, that the input is drawn randomly from some probability distribution μ_n over all inputs of size n.

The avg case time is

$$T_{Avg}(n) = \sum_{|X|=n} \overbrace{Pr[X]}^{\mu_n(X)} \overbrace{T(X)}^{\text{run time on input } X}$$

Weakness: Dist is often not uniform, and one may not even know what μ_n is.

A Randomize Algorithm is one that makes random choices as part of the ~~algoret~~ algorithm.

Analysis is worst-case expected time: Given an input X

$$T_{exp}(X) = \underset{\substack{\uparrow \\ \text{random choices} \\ \text{made by the algorithm}}}{E}[T(X)]$$

Worst-case expected time is

$$E(n) = \max_{|X|=n} T_{exp}(X)$$

~~Pref~~ Preferred analysis for rand. algos.

RandomizedQuicksort$(A[p..q])$

- Choose a pivot x uniformly at random from $A[p], \ldots, A[q]$
- same except recursively call RandomizedQuicksort.

Given a random variables T, U

$$E[T] = \sum_{x \in \mathbb{R}} x \Pr[T = x]$$

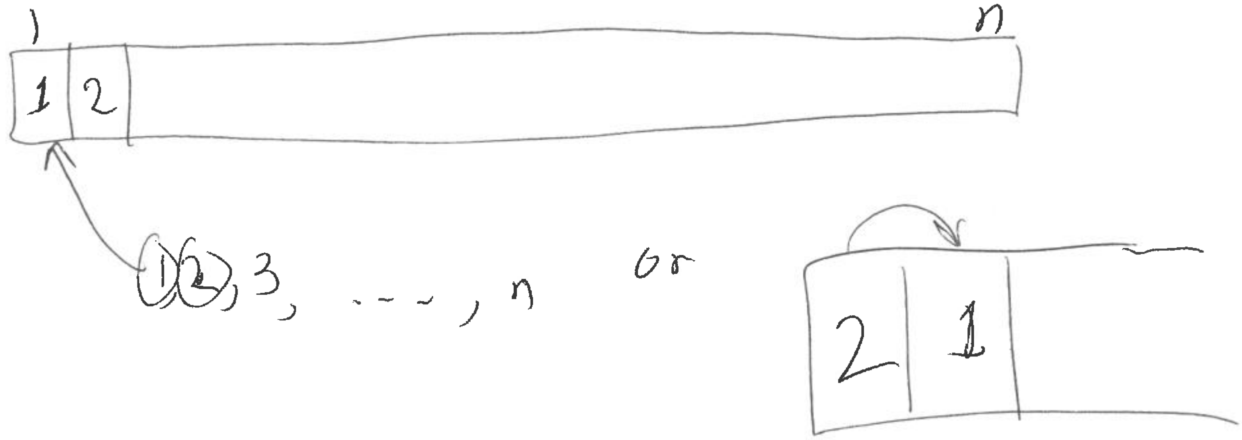
Properties

$$E[T+U] = E[T] + E[U]$$

$(c \in \mathbb{R}$ constant$)$ $$E[cT] = c\,E[T]$$

linearity of expectation

<u>Example</u>: Generating a random permutation of $1, \ldots, n$



RandomPerm(n)

$\Theta(1)$ — Allocate A[1..n]

for i := 1 to n do:

// insert i randomly into A[1..i]

// bumping values to the right as necessary

$\Theta(1)$ — let r := uniformly randomly chosen integer in the range [1..i]

$\Theta(i-r)$ — for j := i downto r+1

A[j] := A[j-1]

$\Theta(1)$ — A[r] := i

Expected run time is

$$E(n) = \sum_{i=1}^{n} E[\text{time of } i\text{th iteration}]$$

$$= \sum_{i=1}^{n} \sum_{r=1}^{i} (\text{Time given } r) \cdot \Pr[r]$$

$\Pr[R=r]$

$$= \Theta\left(\sum_{i=1}^{n} \sum_{r=1}^{i} (i-r) \cdot \frac{1}{i} \right)$$

$$= \Theta\left(\sum_{i=1}^{n} \sum_{r=1}^{i} \left(1 - \frac{r}{i}\right) \right)$$

$$= \Theta\left(\sum_{i=1}^{n} \left(i - \frac{1}{i} \sum_{r=1}^{i} r \right) \right)$$

$$= \Theta\left(\sum_{i=1}^{n} \left(i - \frac{1}{\cancel{i}} \left(\frac{\cancel{i}(i+1)}{2} \right) \right) \right)$$

$$= \Theta\left(\sum_{i=1}^{n} \left(i - \frac{(i+1)}{2} \right) \right)$$

$$= \Theta\left(\sum_{i=1}^{n} \left(\frac{i}{2} - \frac{1}{2} \right) \right)$$

$$= \Theta\left(\frac{1}{2} \sum_{i=1}^{n} i - \frac{1}{2} \sum_{i=1}^{n} 1 \right)$$

$$= \Theta\left(n^2 \right)$$

Analysis of RandomizedQuicksort

Choosing pivot & partitioning — $\Theta(n)$ time

right list has size $n-q-1$

Let q be the size of the left list (to the left of the pivot)

Assuming no duplicates

Then $0 \le q \le n-1$ uniformly random

Let $E(n)$ be the expected time of RandomizedQuicksort for n items

(this is actually independent of the input arrangement

$$E(n) = \underbrace{\Theta(n)}_{\text{partition \& pivot choice time}} + \sum_{q=0}^{n-1} \overbrace{\Pr[q]}^{1/n} \cdot \left(\underbrace{E(q)}_{\substack{\text{expected time to sort}\\ \text{left list, assuming size } q}} + \underbrace{E(n-q-1)}_{\substack{\text{right list}\\ \text{size } n-q-1}} \right)$$

$$= an + \frac{1}{n} \sum_{q=0}^{n-1} \left(E(q) + E(n-q-1) \right)$$

$$E(n) = \boxed{an + \frac{2}{n} \sum_{q=0}^{n-1} E(q)}$$

Verify that $E(n) = O(n \lg n)$ via substitution method

$$\boxed{\text{assume } E(q) = c\,q\lg q}$$

$$E(n) = an + \frac{2}{n}\sum_{q=0}^{n-1} E(q)$$

$$\le an + \frac{2}{n}\sum_{q=0}^{n-1} c\,q\lg q$$

$$= an + \frac{2}{n\ln 2}\sum_{q=0}^{n-1} c\,q\ln q \quad \text{(natural log)}$$

$$\le an + \frac{2c}{n\ln 2}\int_1^n x\ln x\,dx$$

$0 < x < 1$: $x\ln x < 0$

$$= an + \frac{2c}{n\ln 2}\left[\frac{x^2\ln x}{?} + \text{smaller terms}\right]_1^n$$

$$= an + \frac{2c}{n\ln 2}\left(\frac{n^2\ln n}{\text{constant}} + \text{smaller terms}\right)$$

$$= an + \frac{2c}{\text{const}}\,n\ln n$$

$$= an + \frac{2c}{\text{const}}\,n\lg n \overset{?}{\le} cn\lg n$$

notes!