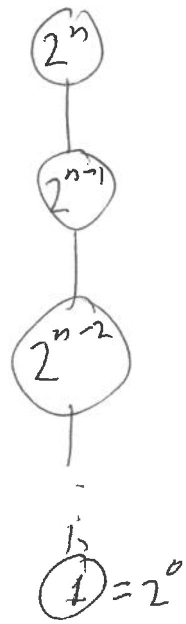
CSCE 750
9/12/2023

$\log_3 n = \Theta(\lg n)$

Tree method

$T(n) = T(n-1) + 2^n$



$2^{n+1} = 2 \cdot 2^n$

$$\sum_{i=0}^{n} 2^i = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1 = \Theta(2^n)$$

Master theorem (simple version)

Let T satisfy the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $f(n)$ is eventually positive.

~~Then~~ Let $s := \log_b a$

~~Cs~~ Then:

Case 1: If $f(n) = O(n^t)$ for some constant $t < s$, then $T(n) = \Theta(n^s)$.

Case 2: If $f(n) = \Theta(n^s)$, then $T(n) = \Theta(n^s \lg n)$

Case 3: If $f(n) = \Omega(n^t)$ for some constant $t > s$ and

~~Eend~~ [regularity condition]

for all sufficiently large n,

$a f\left(\frac{n}{b}\right) \le (1-\varepsilon) f(n)$ for some constant $\varepsilon > 0$,

then

$T(n) = \Theta(f(n))$

The textbook has a stronger Master Theorem (4th ed.)

Applying the master theorem

$$T(n) = 2T\left(\frac{n}{2}\right) + n \qquad f(n) = n$$

$a = 2, \quad b = 2$ $\qquad s := \log_b a = \lg 2 = 1$

$n^s = n^1 = n = f(n)$

Case 2: $T(n) = \Theta(n \lg n)$

$$T(n) = 3T\left(\frac{n}{2}\right) + n \qquad \begin{matrix} b = 2 \\ a = 3 \end{matrix} \qquad s := \lg 3$$

$f(n) = n = n^1$

Case 1: $T(n) = \Theta(n^{\lg 3})$

$$T(n) = 3T\left(\frac{n}{2}\right) + n^2 \qquad 2 > \lg 3$$

Case 3: $T(n) = \Theta(n^2)$.

$$T(n) = 8T\left(\frac{n}{3}\right) + \frac{n^2}{\lg n} \qquad \begin{matrix} a = 8 \\ b = 3 \end{matrix}$$

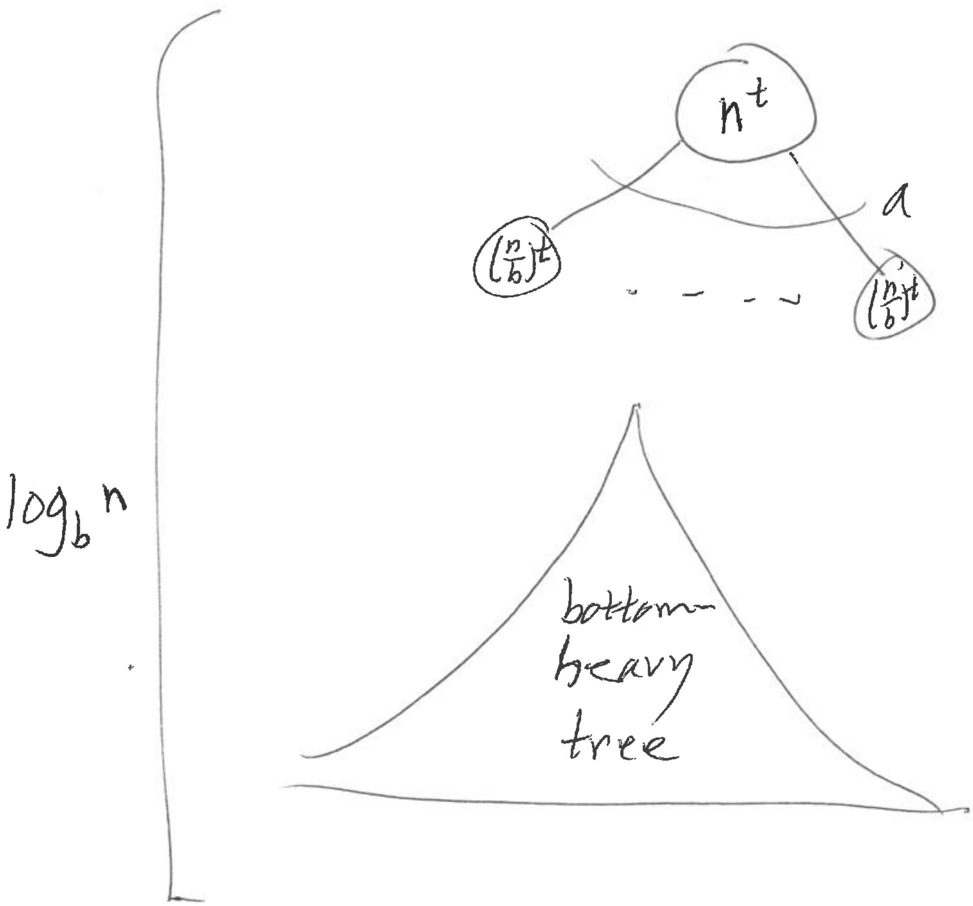
$s = \log_3 8 < 2$

Case 3: $T(n) = \Theta\left(\frac{n^2}{\lg n}\right)$ $\qquad f(n) = \frac{n^2}{\lg n}$

Verifying Master Thm with the tree method

Case 1 : $f(n) = n^t$ $\quad (t < s)$ $\quad s = \log_b a$

$$T(n) = a\,T\left(\frac{n}{b}\right) + \underbrace{f(n)}_{n^t}$$



— n^t

— $a\left(\frac{n}{b}\right)^t = \frac{a}{b^t}\,n^t$

— $\left(\frac{a}{b^t}\right)^2 n^t$

— $\left(\frac{a}{b^t}\right)^3 n^t$

$\vdots$

Note: $t < s \Rightarrow$

$$\frac{a}{b^t} > \frac{a}{b^s} = \frac{a}{b^{\log_b a}} = \frac{a}{a} = 1$$

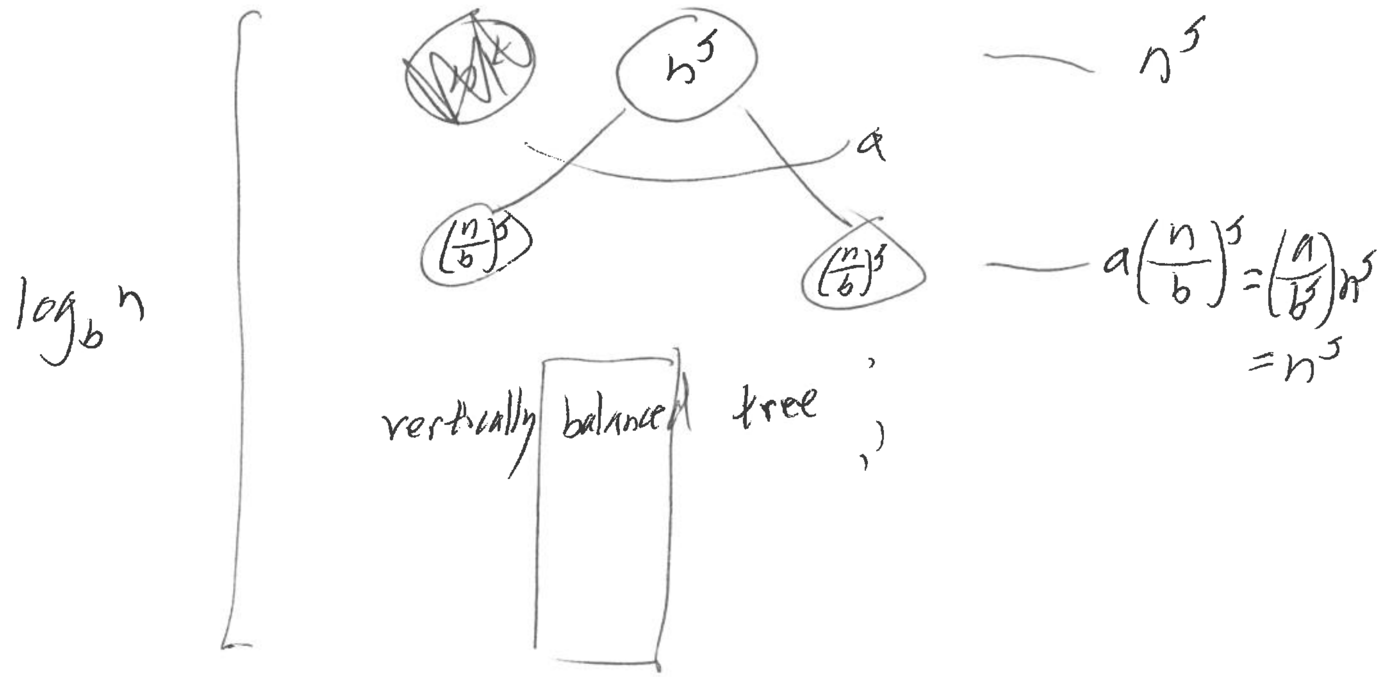
$$T(n) = \sum_{i=0}^{\log_b n} \left(\frac{a}{b^t}\right)^i n^t = n^t \sum_{i=0}^{\log_b n} \left(\frac{a}{b^t}\right)^i$$

$$= n^t \left(\frac{\left(\frac{a}{b^t}\right)^{\log_b n} - 1}{\frac{a}{b^t} - 1} \right)$$

$$= \Theta\left(n^t \left(\frac{a}{b^t}\right)^{\log_b n}\right) = \Theta\left(n^t\, n^{\log_b (a/b^t)}\right)$$

$$= \Theta\left(n^{t + \log_b a - \log_b b^t}\right) = \Theta\left(n^{\log_b a}\right) = \Theta(n^s)$$

Case 2: $f(n) = n^5 = n^{\log_b a}$



$$T(n) = \sum_{i=0}^{\log_b n} \underbrace{\left(\frac{a}{b^5}\right)^i}_{1} n^5$$

$$= n^5(\log_b n + 1)$$

$$= \Theta(n^5 \log_b n)$$

$$= \Theta(n^5 \lg n)$$

Case 3: $f(n) = n^t \quad (t > s)$

same tree as in case 1.

$$T(n) = \sum_{i=0}^{\log_b n} \underbrace{\left(\frac{a}{b^t}\right)}_{<1} n^t$$

$$\leq \sum_{i=0}^{\infty} \left(\frac{a}{b^t}\right) n^t$$

$$= n^t \left(\frac{1}{1 - a/b^t} \right)$$

$$= \Theta(n^t)$$

$T(n) = O(n^t)$

$T(n) \geq n^t$ (just take the root of the tree)

$\therefore T(n) = \Omega(n^t)$

$\therefore T(n) = \Theta(n^t)$

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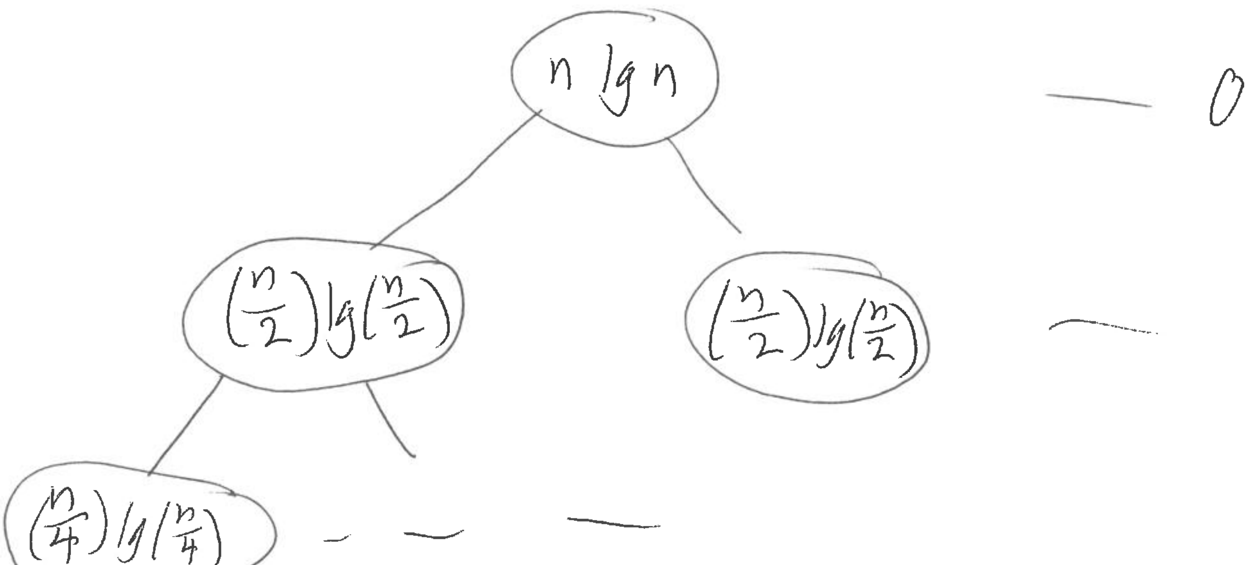
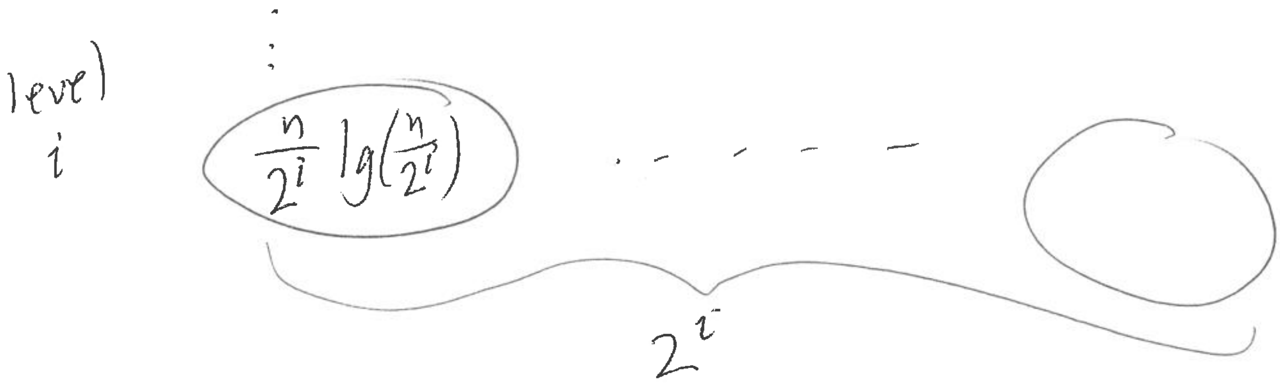
simple M.T. does not apply to
$T(n) = T(n-1) + 2^n$
$T(n) = T\left(\frac{3n}{5}\right) + T\left(\frac{4n}{5}\right) + n^2$
$T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$

Tree method:





Depth $= \lg n$

$$T(n) = \sum_{i=0}^{\lg n} \cancel{2^i}\left(\frac{n}{\cancel{2^i}}\right)\lg\left(\frac{n}{2^i}\right)$$

$$= n\sum_{i=0}^{\lg n} \lg\left(\frac{n}{2^i}\right) = n\sum_{i}\left(\lg n - \lg 2^i\right)$$

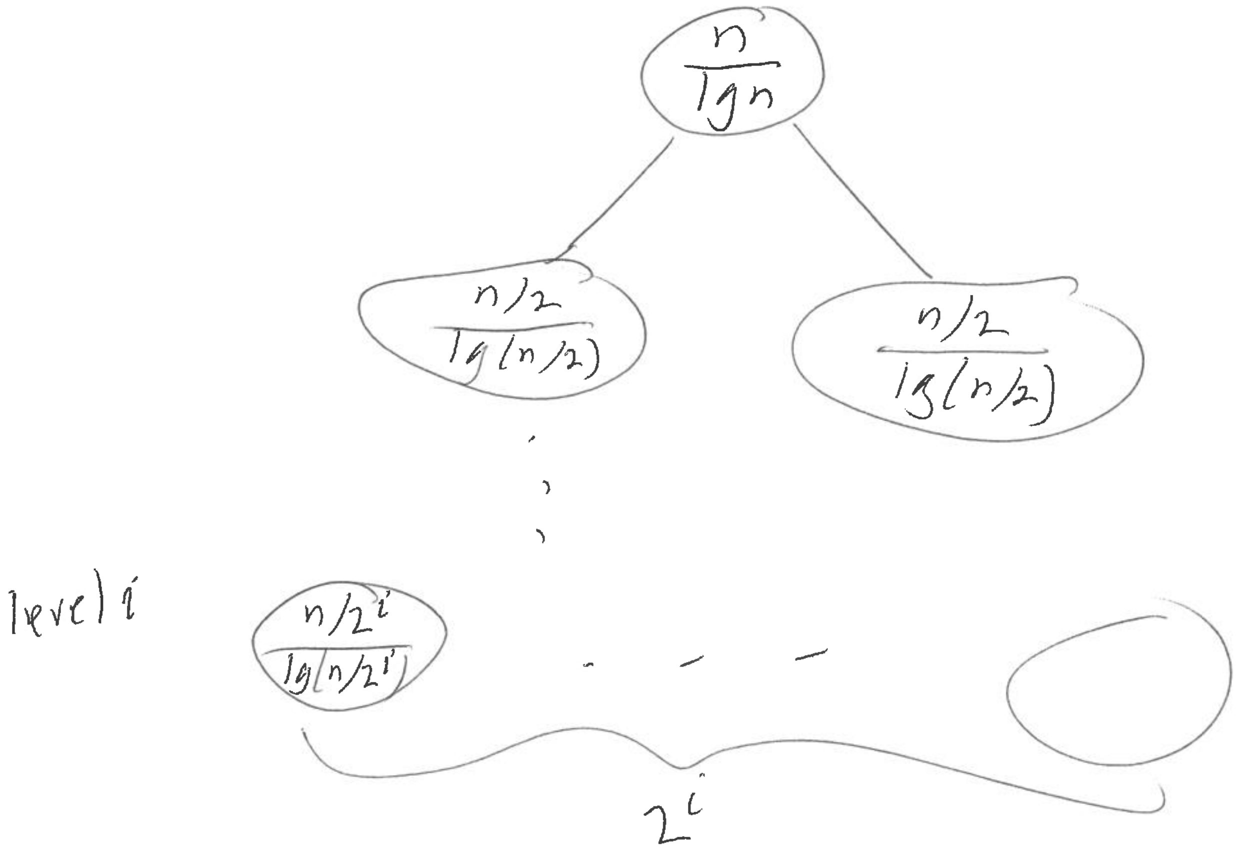
$$= n\sum_{i=0}^{\lg n}\left(\lg n - i\right) = n\left(\sum_{i=0}^{\lg n}\lg n - \sum_{i=0}^{\lg n} i\right)$$

$$= n\left(\underline{(\lg n)^2 + \lg n} - \frac{\lg n(\lg n + 1)}{2}\right)$$

$$= n\left(\frac{\lg n(\lg n + 1)}{2}\right)$$

$$= \Theta\left(n(\lg n)^2\right) = T(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$$



$$\therefore\ T(n) = \sum_{i=0}^{\lg n} \left(\frac{n/2^i}{\lg(n/2^i)}\right) 2^i = n \sum_{i=0}^{\lg n} \frac{1}{\lg(n/2^i)}$$

$$= n \sum_{i=0}^{\lg n - 1} \frac{1}{\lg n - i}$$

$j := \lg n - i$

re-index the sum

$$= n \sum_{j=1}^{\lg n} \frac{1}{j} = \Theta(n \lg(\lg n))$$

$$= \Theta(n \lg \lg n)$$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

Case 3: $T(n) = \Theta(n)$

$s := \lg 1 = 0$

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

Case 2: $T(n) = \Theta(\lg n)$