CSCE 750
8/31/2023

Some functions & their asymptotics

Fixed $k \geq 0$ (any real number ≥ 0)

$f(n) = n^k$ $\qquad k < \ell \Rightarrow n^k = o(n^\ell)$

$f(n) = 2^n$ $\qquad$ exponential function with base 2

$n^k = o(2^n) \quad \forall k$

Thus $\quad 2^n \notin \mathcal{P}oly(n)$

<u>Logarithms</u>: Fix $b > 0$, $b \neq 1$. $\forall x > 0$,

$\log_b x$ is the unique real number y

such that $b^y = x$

$\log_2 2^n = n$

$2^{\log_2 n} = n$

Natural exp fcn

$x \mapsto e^x \qquad (e = 2.71828\ldots)$

$\ln x = \log_e x \qquad (x > 0)$

"take logs" $\qquad \log_2 n = o(n^\varepsilon)$ for any $\varepsilon > 0$

<u>Change of base rule</u>: $\log_a n = \dfrac{\log_b n}{\log_b a}$ $\quad$ ($a, b > 0$, $a \neq 1, b \neq 1$) $\quad a, b > 1$

$\log_a n = \Theta(\log_b n)$

Abbrev.: $\boxed{\lg = \log_2}$

Rules: $\log_b(xy) = \log_b x + \log_b y$ product rule

quotient rule $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$ $\left(\begin{matrix} x, y \in R, \\ x, y > 0 \end{matrix}\right)$

$\therefore \log_b\left(\frac{1}{x}\right) = \log_b 1 - \log_b x = -\log_b x$ $b > 0, b \neq 1$

power rule: $\log_b x^r = r \log_b x$ $(r \in R, x > 0)$

$$\boxed{a^{\log_b c} = c^{\log_b a}}$$

$\log_b$ is one-to-one

$\log_b$ of both sides, use the power rule:

proof

LHS:

$\log_b\left(a^{\log_b c}\right) = \log_b c \log_b a$

$\parallel$

RHS $\log_b\left(c^{\log_b a}\right) = \log_b a \log_b c$

Summations

— finding runtime of iterative programs:

```
int A(int n)
{
    int sum=0;
    int i,j;
    for (i=0; i<n; i++)
        for (j=0; j<n; j++)
```

$\Theta(1)$

```
        sum++;
    return sum;
}
```


$$\sum_{j=\ell}^{u} a_j = a_\ell + a_{\ell+1} + \cdots + a_u$$

$\ell, u \in \mathbb{Z}$ (integers)

[convention: if $u < \ell$, then sum is 0]

Time for inner loop is

$$\sum_{j=0}^{n-1} \Theta(1) = \Theta\left(\underbrace{\sum_{j=0}^{n-1} 1}\right) = \Theta(n)$$

Asymptotic Time for outer loop:

$$\sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-1} 1\right) = \sum_{i=0}^{n-1} n = n^2$$

$\therefore$ Total time is $\Theta(n^2) + \Theta(1) = \Theta(n^2)$

Ex:

```
int B(int n)
{
    int sum=0, i,j;
    for(i=1; i<=n; i++)
        f(j=0; j<i; j++)
            sum++;
    return sum;
}
```

$$\sum_{j=0}^{i-1} 1$$

$$\sum_{j=1}^{n}$$

$$\sum_{i=1}^{n} \sum_{j=0}^{i-1} 1 = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

Arithmetic sum

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \Theta(n^2)$$

squared

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3)$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \Theta(n^4)$$

$$\vdots$$

constant $k \geq 0$

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$$

Proof: (1) upper bound: show that $\sum_{i=1}^{n} i^k = O(n^{k+1})$:

$$\sum_{i=1}^{n} i^k \leq \sum_{i=1}^{n} n^k = n \cdot n^k = n^{k+1} \in O(n^{k+1})$$

lower bd (2)

$$\sum_{i=1}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i^k \geq \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^k$$

$$\geq \left(\frac{n}{2}\right)\left(\frac{n}{2}\right)^k = \frac{n^{k+1}}{2^{k+1}} = c\,n^{k+1}$$

$$= \Omega(n^{k+1})$$

$\left(c = \frac{1}{2^{k+1}} > 0 \text{ constant}\right)$

$$\boxed{\text{Show: } \sum_{i=1}^{n} i^k \in \Omega(n^{k+1})}$$

∎

```
int C(int n)
{
    int sum=0;
    int i,j;
    for(i=1; i<=n; i *=2)
        for(j=0; j<i; j++)
            sum++;
    return sum;
}
```

$\Theta(i)$

call it $i=2^{k-1}$

k'th iteration:

$2^{k-1}=i$

$$\sum_{k:2^{k-1}\le n,\ k\geq 0} \overbrace{2^{k-1}}^{i}$$

integer

u = largest integer k s.t. $2^{k-1}\le n$

i.e., $\lg 2^{k-1} \le \lg n$

$k-1 \le \lg n$

$k \le 1+\lg n$

$k = \lfloor 1+\lg n \rfloor$

$$\text{Total time} = \sum_{k=1}^{u} 2^{k-1}$$

$$= \sum_{k=1}^{1+\lfloor \lg n\rfloor} 2^{k-1}$$

$$(\ell := k-1) = \underbrace{\sum_{\ell=0}^{\lfloor \lg n\rfloor} 2^{\ell}}_{\text{finite geometric sum}} = \frac{2^{1+\lfloor \lg n\rfloor}-1}{2-1}$$

$$= 2^{1+\lfloor \lg n\rfloor}-1$$

$$= 2\cdot 2^{\lfloor \lg n\rfloor}-1$$

$$= \Theta\left(2^{\lfloor \lg n\rfloor}\right)$$

claim

$$= \Theta(n)$$

(6)

$x \in \mathbb{R}$: $\lfloor x \rfloor \in \mathbb{Z}$ and

$$\underline{x-1 < \lfloor x \rfloor \leq x}, \qquad x \leq \lceil x \rceil < x+1$$

$\lceil x \rceil \in \mathbb{Z}$

$$2^{\lfloor \lg n \rfloor} \leq 2^{\lg n} = n$$

$$2^{\lfloor \lg n \rfloor} > 2^{\lg n - 1} = \frac{2^{\lg n}}{2} = \frac{n}{2}$$

$\Theta(n)$

Finite geometric sums in general:

Fix any $r \neq 1$ $(r \in \mathbb{R})$

$$\boxed{\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r-1}}$$

$$S := \sum_{i=0}^{n-1} r^i = 1 + r + r^2 + \cdots + r^{n-1}$$

$j := i+1$

$$rS := \sum_{i=0}^{n-1} r^{i+1} = \sum_{j=1}^{n} r^j = r + r^2 + \cdots + r^{n-1} + r^n$$

$$rS - S = r^n - 1$$

$$rS - S = (r-1)S$$

$$S = \frac{r^n - 1}{r-1} \quad \square$$

For $|r| < 1$, $\displaystyle\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty} \sum_{i=0}^{n-1} r^i = \lim_{n\to\infty} \left(\frac{r^n - 1}{r-1}\right)$

$$= \frac{\left(\lim_{n\to\infty} r^n\right) - 1}{r-1} = \frac{0-1}{r-1} = \frac{-1}{r-1} = \boxed{\frac{1}{1-r}}$$

Harmonic Sums

A harmonic sum is of the form

$$\boxed{\sum_{i=1}^{n} \frac{1}{i} = \Theta(\lg n)}$$

Proof sketch:

$B_i \leq 1$

$$\sum_{i=1}^{n} \frac{1}{i} = \underbrace{\overbrace{1}^{B_0} + \overbrace{\frac{1}{2}}^{B_1}}_{C_0 \; C_1} + \overbrace{\underbrace{\frac{1}{3} + \frac{1}{4}}_{C_2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}^{B_2} + \frac{1}{8} + \cdots + \frac{1}{n}$$

$$C_i \geq \frac{1}{2}$$

roughly $\lg n$ many blocks