CSCE 750
8/29/2023

Asymptotics:

Last time: f, g eventually positive

$f \in O(g)$ $\quad [f(n) = O(g(n))]$

means $\exists C > 0, \exists n_0 \; \forall n \geq n_0, f(n) \leq C g(n)$

$f(n) = O(g(n))$ is like "$\leq$" ignoring some "slop"

$f(n)$ is "asymptotically" $\leq g(n)$

$f \in \Omega(g)$ means $g \in O(f)$

equiv. $\exists C > 0 \; \exists n_0 \; \forall n \geq n_0, f(n) \geq C g(n)$

Ω is like $\geq$

Reflexivity $\quad f \in O(f)$

$f \in \Omega(f)$

Transitivity $\quad$ If $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$.

Proof: Let $n_1, C_1 > 0$ be such that $\forall n \geq n_1$
$$f(n) \leq C_1 g(n)$$

Let $n_2, C_2 > 0$ be such that $\forall n \geq n_2$
$$g(n) \leq C_2 h(n)$$

~~Let~~ To show $f \in O(h)$,

let $n_0 := \max(n_1, n_2)$,

let $C := c_1 c_2$

$\forall n \geq n_0,\quad f(n) \leq c_1 g(n) \leq c_1(c_2 h(n)) = c_1 c_2 h(n) = C h(n)$ $\blacksquare$

$\therefore f \in O(h)$ via n_0 and C.

Def: $f \in \Theta(g)$ means $f \in O(g)$ and $\underbrace{f \in \Omega(g)}_{g \in O(f)}$

"f & g are "asymptotically similar""

$f \in \Theta(g)$ is an equivalence relation.

"Tight asymptotic bounds" on a function $f(n)$ means find a $g(n)$ (simple as possible) such that $f \in \Theta(g)$.

Ex: $f(n) = 3n^5 - 6n^2 + 9 \in \Theta(n^5)$

Little-o notation:

$f \in o(g)$ [$f(n) = o(g(n))$} means

$\forall c > 0, \exists n_0, \forall n \geq n_0,\ f(n) \leq c\,g(n)$

[equivalently, $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$]

f grows strictly slower than g

Little-ω notation:

$f \in \omega(g)$ means $g \in o(f)$.

f grows strictly faster than g

Equivalently,

$$\forall C > 0, \exists n_0 \, \forall n \geq n_0, \; f(n) \geq C g(n)$$

$$\left[\text{equiv: } \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \right]$$

Facts:

$f \in o(g) \Rightarrow f \in O(g)$

$f \in \omega(g) \Rightarrow f \in \Omega(g)$

$f \in o(g) \Rightarrow f \notin \Theta(g)$

$f \in \omega(g) \Rightarrow f \notin \Theta(g)$

True or false: $f \in o(g) \Leftrightarrow f \in O(g) \;\&\; f \notin \Theta(g)$

False; counterexample: $g(n) = n$

$$f(n) = \begin{cases} n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$f \notin \Omega(g)$: $\forall C > 0 \; \forall n_0$, let n be the least odd number $> \max(\frac{1}{C}, n_0)$ ~~such th~~

$n > n_0$. $\quad f(n) = 1$

$$g(n) = n > \frac{1}{C} \Rightarrow C g(n) > C \cdot \frac{1}{C} = 1 = f(n)$$

$\therefore f(n) \ngeq C g(n)$ for this chosen n. $\therefore f \notin \Omega(g)$.

And $f \in O(g)$: let $n_0 := 1$ and $c := 1$.

Then $f(n) \le n = c\,g(n) \quad \forall n \ge n_0$

& $f \notin \Theta(g)$ [because $f \notin \Omega(g)$]

but $\lim\limits_{n \to \infty} \frac{f(n)}{g(n)}$ does not exist $\left(\frac{f(n)}{g(n)} = \begin{cases} 1 & \text{even } n \\ \frac{2}{n} & \text{for odd } n \end{cases} \right)$

$\therefore f \notin o(g)$

Def: f is monotone ascending if $\forall n_1, n_2$,

$$n_1 \le n_2 \Rightarrow f(n_1) \le f(n_2)$$

f is strictly monotone increasing if $\forall n_1, n_2$

$$n_1 < n_2 \Rightarrow f(n_1) < f(n_2)$$

Puzzle: Find monotone ascending f, g such that $f \in O(g)$, $f \notin \Omega(g)$, and $f \notin o(g)$

Technically, $O(g)$, $\Theta(g)$, $o(g)$, $\omega(g)$, $\Omega(g)$ are classes of functions

$f(n) = O(g(n))$ means $f \in O(g)$

similarly for the other classes.

If A, B are sets of functions, define

$$A+B = \{f+g : f \in A \;\&\; g \in B\}$$

$$AB = \{fg : \text{''} \quad \text{''}\}$$

$$A^B = \{f^g : \ldots\}$$

$$f + B := \{f\} + B = \{f + g : g \in B\}$$

Ex: $n^2 + \Theta(n) \underset{\not\supseteq}{\subseteq} \Theta(n^2)$

$\overset{?}{\supseteq}$

$2n^2 \in \Theta(n^2)$

$\left.\begin{array}{l} 2n^2 \overset{?}{\in} n^2 + \Theta(n) \end{array}\right]$ false!

$\quad \shortparallel$

$n^2 + n^2$

$\Theta(1)$ functions bounded by constants

$n + \Theta(1) \underset{\not\supseteq}{\subseteq} \Theta(n)$

$$n^{\Theta(1)} = \bigcup_{k \geq 0} \Theta(n^k) =: \mathrm{Poly}(n)$$