2 1 CSCE 355
9/1/2022

Complement & product constructions for DFAs
ε-transitions & ε-NFAs

Notation: Fix alphabet Σ. If $A \subseteq \Sigma^*$ is a language, we let
$\overline{A}$ (complement of A): $\overline{A} := \Sigma^* \setminus A = \{w \in \Sigma^* : w \notin A\}$

Theorem: If $A \subseteq \Sigma^*$ is regular, then $\overline{A}$ is regular.

Proof: By construction. Assume A is regular. Let $D := \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA recognizing A $(A = L(D))$.

Def: $\neg D$ (the complement of D) is defined as

$$\neg D := \langle Q, \Sigma, \delta, q_0, Q \setminus F \rangle$$ [swap rejecting $\Leftrightarrow$ accepting]

Claim: $L(\neg D) = \overline{L(D)} = \overline{A}$.

Proof of claim: Let $w \in \Sigma^*$.

$w \in L(\neg D) \Leftrightarrow \neg D$ accepts $w \Leftrightarrow \hat{\delta}(q_0, w) \in Q \setminus F$

$\Leftrightarrow \hat{\delta}(q_0, w) \notin F \Leftrightarrow D$ rejects w

$\Leftrightarrow w \in \overline{L(D)}$. // claim

$\therefore L(\neg D) = \overline{L(D)}$

(2) ∴ $\overline{A} = L(\neg D)$ ∴ $\overline{A}$ is regular ∎ Thm

[sets S, T $S \setminus T = S - T = \{x \in S : x \notin T\}$

<u>Thm</u>: If $A, B \subseteq \Sigma^*$ are both regular, then $A \cap B$ is regular.

<u>Proof</u>: By construction ("product construction")

Assume A, B regular. Let

$$D_A := \langle Q_A, \Sigma, \delta_A, q_{0,A}, F_A \rangle$$

$$D_B := \langle Q_B, \Sigma, \delta_B, q_{0,B}, F_B \rangle$$

be DFAs such that $A = L(D_A)$ and $B = L(D_B)$.

Define the <u>product</u> DFA $D_A \wedge D_B$ as follows:

$$D := D_A \wedge D_B := \langle Q_A \times Q_B, \Sigma, \delta, (q_{0,A}, q_{0,B}), F_A \times F_B \rangle$$

where, for every $a \in \Sigma$ and states $q \in Q_A$ and $r \in Q_B$

(3) $$\delta((q,r),a) := (\delta_A(q,a), \delta_B(r,a)).$$

Claim: $L(D) = L(D_A) \cap L(D_B)$ $(= A \cap B)$.

Pf of claim: By string induction. Let $w \in \Sigma^*$ be arbitrary.

~~Base case: If $w = \varepsilon$ Next~~ (Need to show $w \in L(D) \Longleftrightarrow w \in L(D_A)$ & $w \in L(D_B)$)

Base case: If $w = \varepsilon$, then

$$\cancel{\hat{\delta}(w)} \quad \hat{\delta}(\underbrace{(q_{0,A}, q_{0,B})}_{\text{start state of } D}, w) = \hat{\delta}((q_{0,A}, q_{0,B}), \varepsilon) = (q_{0,A}, q_{0,B})$$

$$\therefore \underset{\varepsilon}{\overset{\shortparallel}{w}} \in L(D) \Longleftrightarrow \hat{\delta}((q_{0,A}, q_{0,B}), \varepsilon) \in F_A \times F_B$$

$$\Longleftrightarrow (q_{0,A}, q_{0,B}) \in F_A \times F_B \Longleftrightarrow q_{0,A} \in F_A \text{ and } q_{0,B} \in F_B$$

$$\Longleftrightarrow \varepsilon \in L(D_A) \text{ and } \varepsilon \in L(D_B) \Longleftrightarrow w \in L(D_A) \cap L(D_B)$$

By induction prove that $\hat{\delta}((q_{0,A}, q_{0,B}), w) = (\hat{\delta}_A(q_{0,A}, w), \hat{\delta}_B(q_{0,B}, w))$

Base case ✓

(5) $\Longleftrightarrow \hat{\delta}(q$ ...

(4) <u>Inductive case:</u> ~~wa~~ $w \neq \varepsilon$. $w = xa$ $(x \in \Sigma^*, a \in \Sigma)$

Assume (ind hyp) that $\hat{\delta}((q_{0,A}, q_{0,B}), x) = (\hat{\delta}_A(q_{0,A}, x), \hat{\delta}_B(q_{0,B}, x))$.

Then $q_0 := (q_{0,A}, q_{0,B})$

$$\hat{\delta}(q_0, w) \underset{\uparrow}{=} \delta(\underline{\hat{\delta}(q_0, x)}, a) \underset{\uparrow}{=} \delta((\hat{\delta}_A(q_{0,A}, x), \hat{\delta}_B(q_{0,B}, x)), a)$$

ind def of $\hat{\delta}$ $\qquad$ ind hyp

F

$$\underset{\uparrow}{=} (\underline{\delta_A(\hat{\delta}_A(q_{0,A}, x), a)}, \underline{\delta_B(\hat{\delta}_B(q_{0,B}, x), a)})$$

def of δ

$$\underset{\uparrow}{=} (\hat{\delta}_A(q_{0,A}, xa), \hat{\delta}_B(q_{0,B}, xa)) \underset{\uparrow}{=} (\underline{\hat{\delta}_A(q_{0,A}, w), \hat{\delta}_B(q_{0,B}, w)})$$

ind defs of $\hat{\delta}_A$ and $\hat{\delta}_B$ $\qquad$ $w = xa$ $\qquad$ what we wanted.

<u>Now</u>: for any w: $w \in L(D) \Longleftrightarrow \hat{\delta}(q_0, w) \in F_A \times F_B$

$$\underset{\uparrow}{\Longleftrightarrow} (\hat{\delta}_A(q_{0,A}, w), \hat{\delta}_B(q_{0,B}, w)) \in F_A \times F_B$$

by what we just showed

(5) $\iff \hat{\delta}_A(q_{0,A}, w) \in F_A$ and $\hat{\delta}_B(q_{0,B}, w) \in F_B$

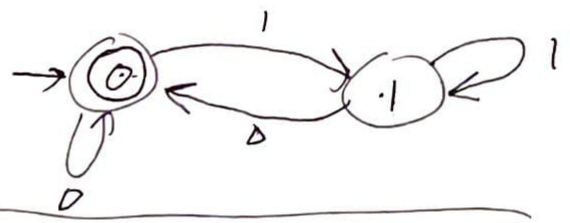
def of cartesian prod

$\iff D_A$ accepts w and D_B accepts $w \iff w \in L(D_A) \cap L(D_B)$ $\blacksquare$

<u>Ex</u>: $\Sigma = \{0,1\}$. $A := \{w \in \Sigma^* : w$ is a mult of 3 in binary$\}$

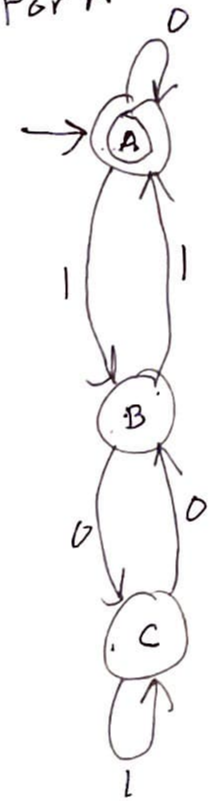
$B := \{w \in \Sigma^* : w$ ~~ex~~ does not end in $1\}$

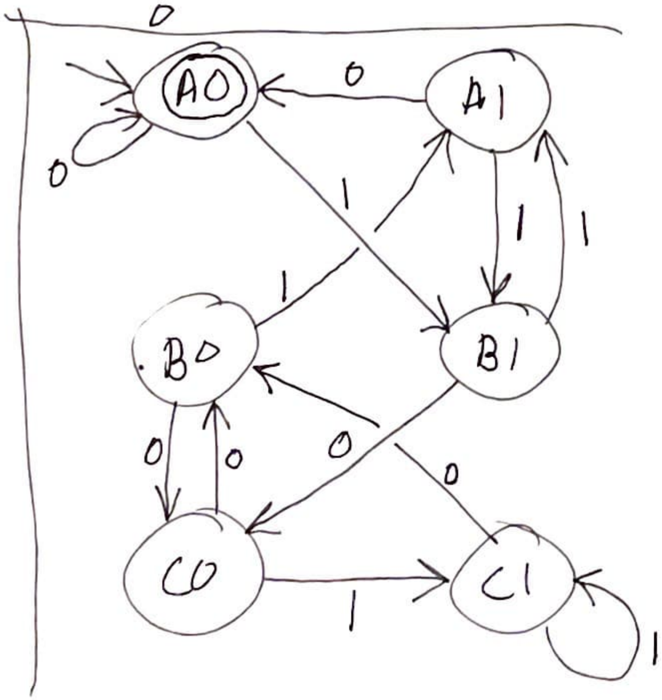
For B



Product construction for $A \cap B$

For A





(6) ε - transitions.

Notation: For alphabet Σ, define $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$ (strings of length 0 or 1).

A ε-transition: $\quad q \xrightarrow{\varepsilon} r \quad$ can move from q to r without consuming a symbol.

An ε-NFA is an NFA where ε-transitions are allowed.

Formally:

Def: An ε-NFA is a tuple $\langle Q, \Sigma, \delta, q_0, F \rangle$ where everything is as with an NFA except

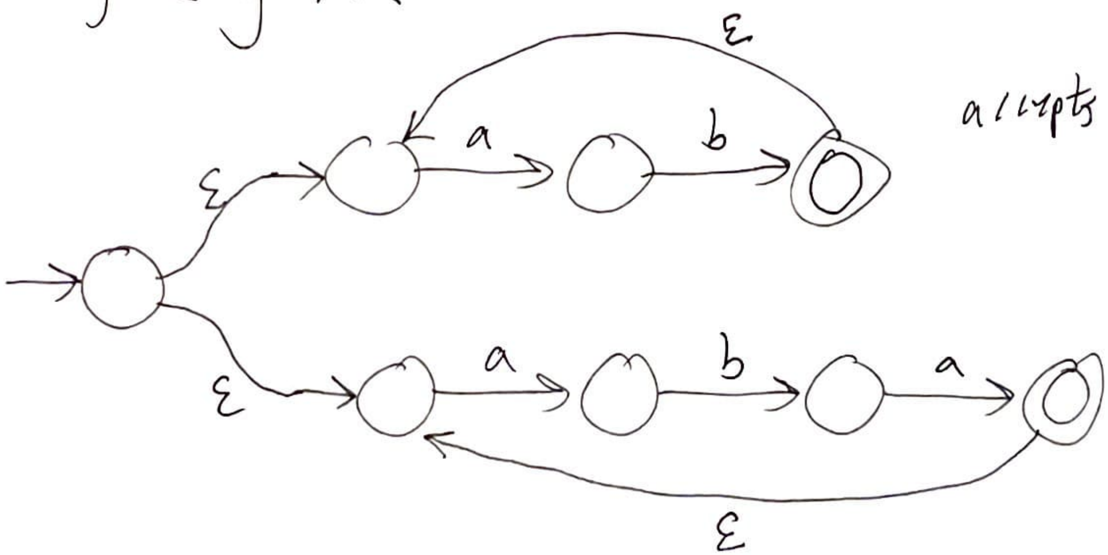
$$\delta : Q \times \Sigma_\varepsilon \longrightarrow 2^Q$$

so

$q \xrightarrow{\varepsilon} r$ means $r \in \delta(q, \varepsilon)$

Ex: $A = \{ w \in \{a,b\}^* :$ w is either ab repeated 1 or more times or w is aba repeated one or more times $\}$

ε-NFA recognizing A:



accepts $(ab)^n$ $(n \geq 1)$

accepts $(aba)^n$ $(n \geq 1)$

Theorem: For any ε-NFA, there exists an equivalent NFA [without ε-transitions].

["equivalent" means recognizing the same language]