① Proof (of the Corollary):

CSCE 355
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Suppose P accepts string $w \in T^*$ (via empty stack)
Then there is a computation

$$\underbrace{(q, w, S)}_{ID_0} \vdash \cdots \vdash \underbrace{(q, \varepsilon, \varepsilon)}_{ID_k}$$

consumed: $x_0 = \varepsilon$ ... $x_k = w$

unconsumed: $y_0 = w$ ... $y_k = \varepsilon$

By the lemma for ID_k there is a sentential form α, derivable from S ($S \Rightarrow^* \alpha$)

where $\alpha = \underbrace{x_k}_{w}\underbrace{(\text{stack contents at step } k)}_{\varepsilon} = w$

so $S \Rightarrow^* w$ $\therefore w \in L(G)$ $\boxed{//}$ corollary.

Lemma: Let $G = \langle V, T, S, P \rangle$ be a CFG and let P be the 1-state PDA constructed before. Let $w \in T^*$ be any string. Assume $w \in L(G)$. Let α be any intermediate sentential form in a leftmost derivation of w, and let $\alpha = xA\beta$ where $x \in T^*$, $A \in V$ and $\beta \in (T \cup V)^*$ [unique decomp:

② because A is the leftmost nonterminal in α.

Then there is a computation of P on input w (not nec. complete) of the form

$$ID_0 = (q, w, S) \vdash \cdots \vdash (q, y, A\beta) = ID_k \quad \begin{pmatrix}\text{some}\\ k\end{pmatrix}$$

where y is such that $\underline{w = xy}$.

[y is the unconsumed portion, so x is the consumed portion of w.]

<u>Proof</u>: By induction on the $\overbrace{\text{number of steps}}^{n}$ to get to α in the leftmost derivation.

$$S = \alpha_0 \Rightarrow \alpha_1 \Rightarrow \cdots \Rightarrow \alpha_n = \alpha$$

$\left[\begin{array}{l} \alpha_i = x_i A_i \beta_i \\ x_i \in T^* \\ A_i \in V \end{array}\right]$

$n = 0$: $\alpha_0 = \underline{S}$

$x_0 = \varepsilon$

$\beta_0 = \varepsilon$

$ID_0 = (q, w, S)$ and $w = \varepsilon w = x_0 w$ ✓

Assume true for α_n, prove true for α_{n+1}
[assumed intermediate sentential form]

By the ind. hyp., there is a computation

$$ID_0 \vdash \cdots \vdash ID_j = (q, y_n, A_n \beta_n) \text{ and } \alpha_n = x_n A_n \beta_n$$

$$w = x_n y_n.$$

③ $\alpha_n \Rightarrow \alpha_{n+1} = x_n \gamma \beta_n$ where $A_n \to \gamma$ is a production of G_1

$\quad \| $

$x_n \underline{A_n} \beta_n$

we have

$$ID_j = (q, y_n, A_n \beta_n) \qquad w = x_n y_n$$

$$ID_j \vdash ID_{j+1} = (q, y_n, \underline{\gamma \beta_n}) \vdash \text{matching}$$ WTS

$$\vdash (q, y_{n+1}, \cancel{A_{n+1}\beta_{n+1}}\ A_{n+1} \beta_{n+1})$$

Let A_{n+1} be the leftmost nonterminal in α_{n+1}

$$\gamma \beta_n = \alpha_{n+1} = \underline{x_{n+1}} A_{n+1} \beta_{n+1}$$

x_{n+1} is a prefix of w

because $\alpha_{n+1} \Rightarrow^* w$

$\underline{x_n}$ is a prefix of $\underline{x_{n+1}}$

consumed at ID_j

matching steps to consume rest of x_{n+1}:

$$ID_j = (q, \cancel{x_n}\, y_n, A_n \beta_n) \vdash (q, y_n, \gamma \beta_n)$$

$$= (q, y_n, \underline{x_{n+1}} A_{n+1} \beta_{n+1})$$ match x_{n+1} against the input

④ $\vdash \cdots \vdash (q, y_{n+1}, A_{n+1}\beta_{n+1})$

where $w = x_{n+1}y_{n+1}$ and $\alpha_{n+1} = x_{n+1}A_{n+1}\beta_{n+1}$. $\boxed{}$

The very last step of the derivation goes from $xAy \Rightarrow xzy = w$

($A \to z$ is a production)

$(q, zy, Ay) \vdash (q, zy, zy)$

$\vdash \cdots \vdash (q, \varepsilon, \varepsilon)$

matching steps

Cor: If $w \in L(G)$ then P accepts w via empty stack.

Pf: Use the Lemma to handle the intermediate steps of the leftmost deriv. of w. Then the last step is handled as above. //

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3 topics to come related to CFLs:

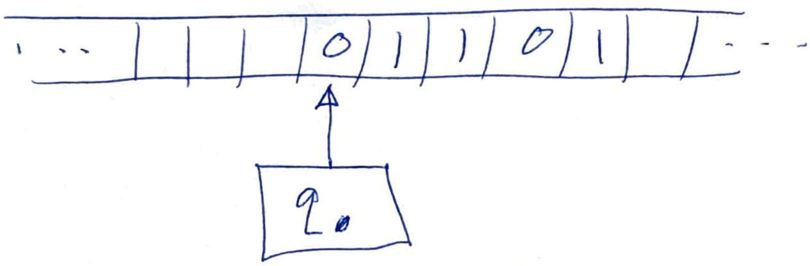
1. PDA $\to$ grammar
2. closure properties of CFLs:
3. Pumping Lemma for CFLs

⑤ Turing Machines (Alan Turing)

Input is on an infinite tape made up of discrete cells.

Ex: $w = 01101$

Initially:



Given state q and symbol a being scanned, the "machine" can

determined by the transition function:

- change state (or not)
- overwrite the cell with another symbol (or not)
- move one cell to the right or left