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Semantics of PDAs (cont.)

CSCE 355
3/16/2022

Recall: PDA $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$

An ID of P is a triple (q, w, γ)

$q \in Q$ current state

$w \in \Sigma^*$ unread portion of the input

$\gamma \in \Gamma^*$ current stack contents (left = top)

Let ID_1 and ID_2 be IDs of P

Define $ID_1 \vdash ID_2$ as follows:

For any $q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, and any $X \in \Gamma$

If $\delta(q, a, X) \ni (r, \beta)$ (contains) then $\boxed{r \in Q \quad \beta \in \Gamma^*}$

$$(q, aw, X\alpha) \vdash (r, w, \beta\alpha)$$

for all $w \in \Sigma^*$ and $\alpha \in \Gamma^*$

Initial ID of P on input w is (q_0, w, Z_0)

Say $ID \vdash^* ID'$ to mean there exist IDs $ID_0, ID_1, \ldots, ID_n$ such that $ID = ID_0 \vdash ID_1 \vdash \cdots \vdash ID_n = ID'$ $(n \geq 0)$

②

Def (Acceptance ~~via~~ by final state):

Let $P = \langle \ldots \rangle$ be a PDA and $w \in \Sigma^*$ an input string. Say that <u>P accepts w via final state</u> if

$$(q_0, w, Z_0) \vdash^* (q, \varepsilon, \alpha)$$

~~where~~ for some $q \in F$ (and $\alpha \in \Gamma^*$ is arbitrary)

Let $L(P) := \{ w \in \Sigma^* : P \text{ accepts } w \text{ via final state} \}$

Def: P as before, $w \in \Sigma^*$ as before. Say that <u>P accepts w via empty stack</u> if

$$(q_0, w, Z_0) \vdash^* (r, \varepsilon, \varepsilon) \text{ for some } r \in Q.$$

Let $N(P) := \{ w \in \Sigma^* : P \text{ accepts } w \text{ via empty stack} \}$

These were the same for the examples given on Monday 3/14, but they are not the same in general

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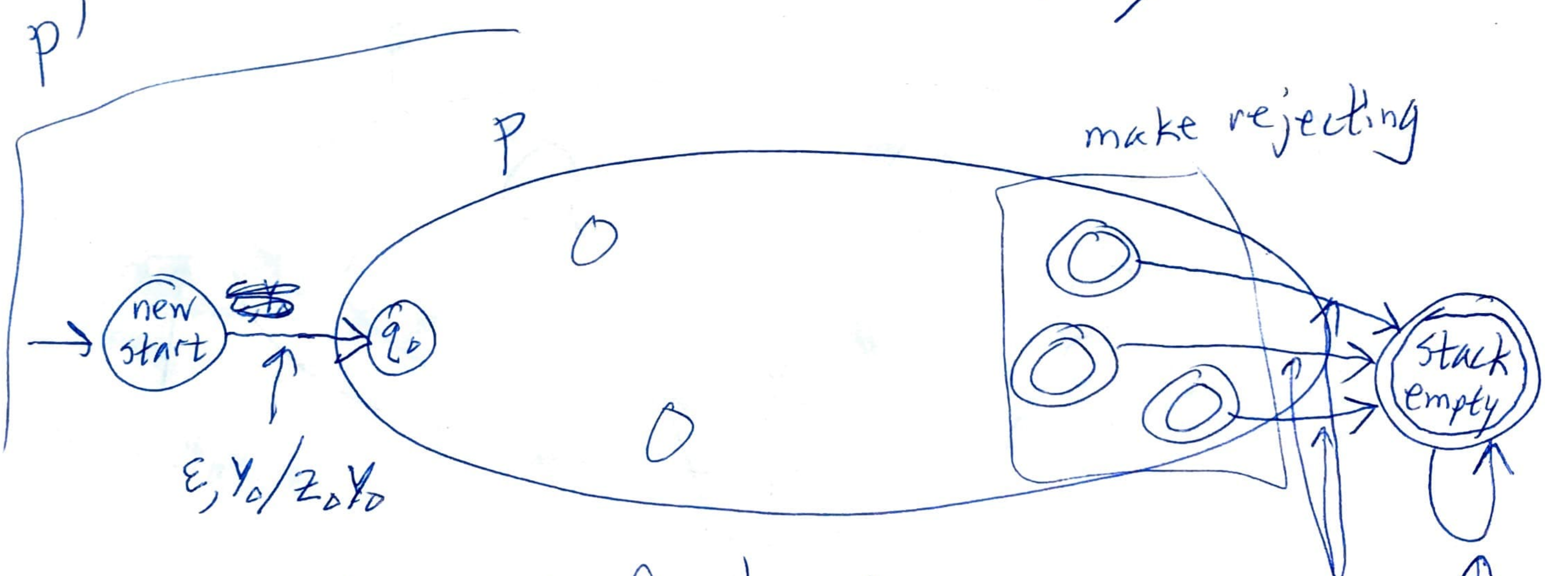
Thm 1: For every PDA P there exists a PDA P' such that $L(P) = N(P')$.

Thm 2: For every PDA P there exists a PDA P' such that $N(P) = L(P')$

③ <u>Proof of Thm 1</u>: Let $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be any PDA. Construct P' as follows:

$$P' = \Big\langle \underbrace{Q \cup \{\text{new start}, \text{stack empty}\}}_{\text{new states}}, \Sigma, \underbrace{\Gamma \cup \{Y_0\}}_{Y_0 \notin \Gamma}, \delta', \text{new start}, Y_0, \{\text{stack empty}\} \Big\rangle$$



$\varepsilon, X/\varepsilon$ for all $X \in \Gamma \cup \{Y_0\}$

If P accepts w via final state, P' accepts w " empty stack ✓ (by the final self-loop)

Conversely, P' can only accept w via empty stack if P accepts w by final state.

$\varepsilon, X/\varepsilon$ for all $X \in \Gamma \cup \{Y_0\}$

$\therefore N(P') = L(P) \; (= L(P'))$

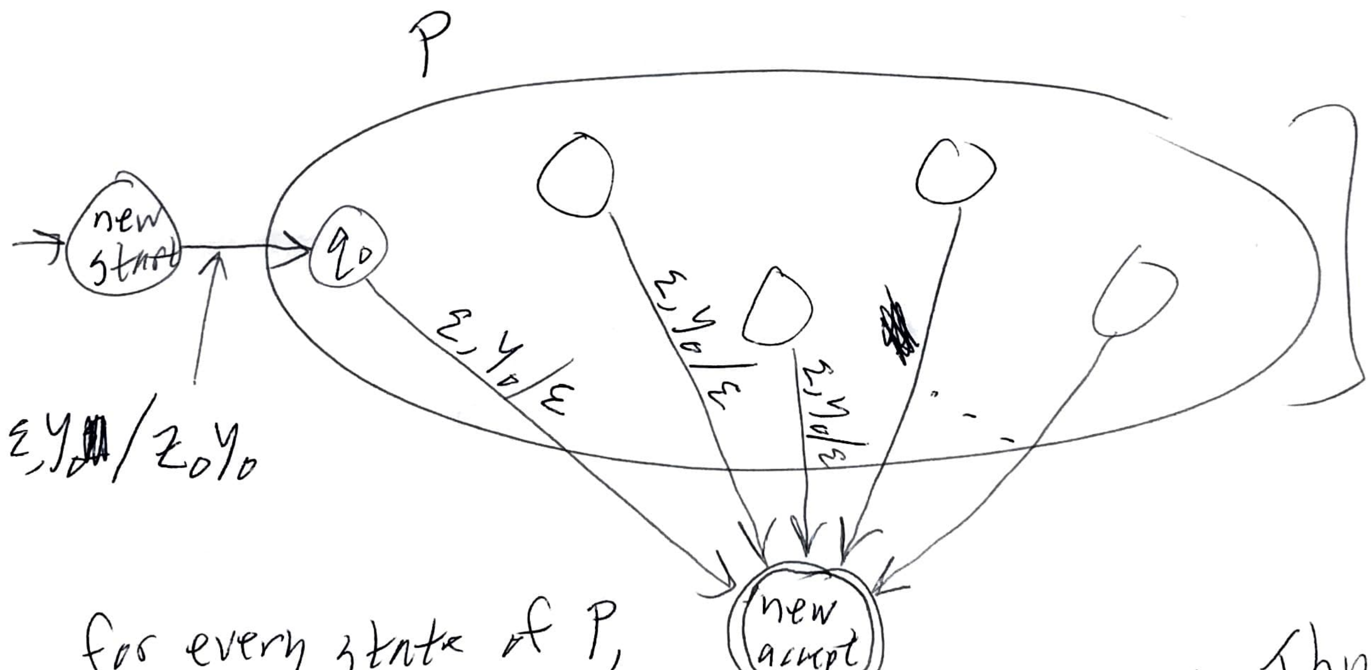
$\boxed{/\!/\!/}$

④ Proof of Thm 2: $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ as before. Construct P' such that $N(P) = L(P')$ as follows:

$$P' = \langle Q \cup \{\text{new start}, \text{new accept}\}, \Sigma, \Gamma \cup \{Y_0\}, \delta', \text{new start}, Y_0, \{\text{new accept}\} \rangle$$

$Y_0 \notin \Gamma$

where δ' is as follows:



for every state of P, add a $\varepsilon, Y_0/\varepsilon$ transition to new accept

P accepts w via empty stack iff P' accepts w via final state [iff P' " " " empty stack]

Thus $N(P) = L(P')$ [$= N(P')$]

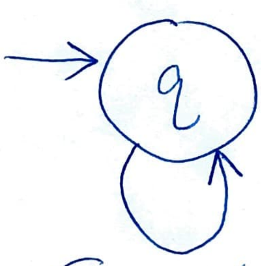
▨

<u>Thm</u>: Let $L \subseteq \Sigma^*$ be any CFL. Then there exists a PDA P (with only one state!) such that $L = N(P)$.

<u>Proof idea</u>: Suppose G is a CFG

<u>Ex</u>: $S \to 0S1 \mid \varepsilon$

PDA P



expansion transitions $\left[\begin{array}{l} \varepsilon, S/0S1 \\ \varepsilon, S/\varepsilon \end{array}\right.$

matching transitions $\left[\begin{array}{l} 0,0/\varepsilon \\ 1,1/\varepsilon \end{array}\right.$

$\Sigma = \{0,1\}$

$\Gamma = \underbrace{\{0,1,S\}}_{V \cup T}$

stack bottom marker = start symbol

<u>Ex input</u>: 000111

Accepting computational trace:

$(q, 000111, S) \vdash (q, 000111, 0S1) \vdash (q, 00111, S1)$
$\vdash (q, 00111, 0S11) \vdash (q, 0111, S11) \vdash (q, 0111, 0S111)$
$\vdash (q, 111, S111) \vdash (q, 111, 111) \vdash (q, 11, 11) \vdash (q, 1, 1) \vdash (q, \varepsilon, \varepsilon)$