① **Pumping Lemma**: Every regular language is ~~pumple~~ pumpable.

CSCE 355
2/23/22

Use: Show a lang not regular by showing that it is not pumpable.

Recall: L is pumpable iff

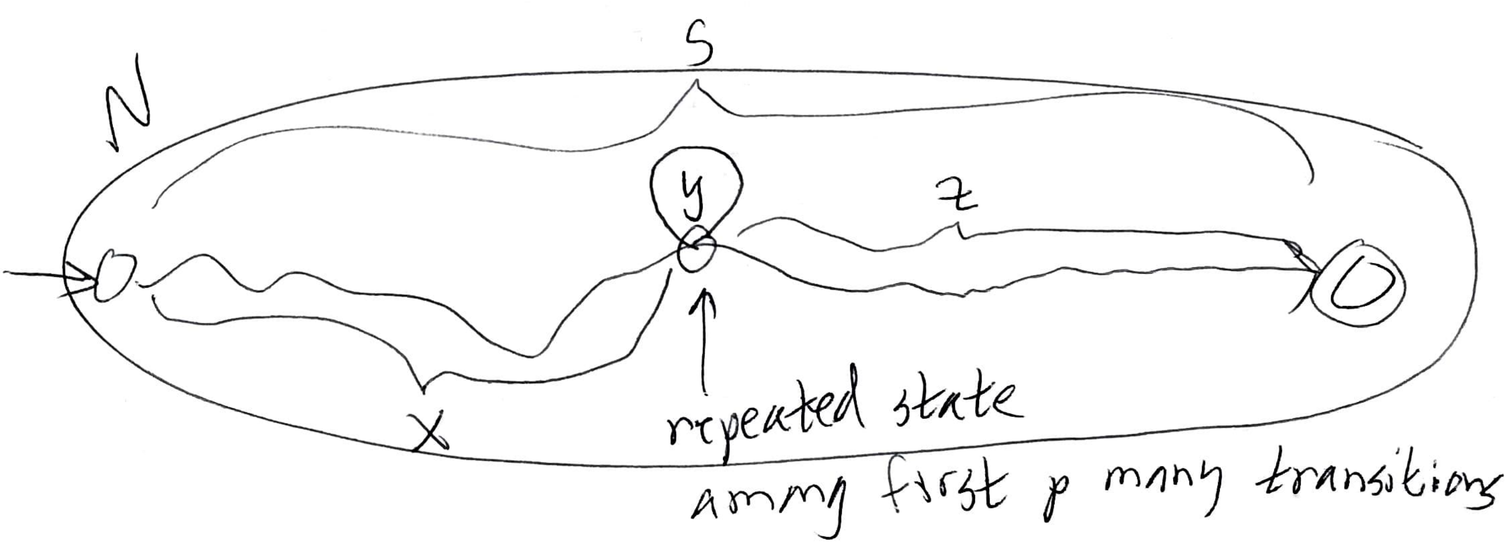
$\exists p > 0,$

$\forall s \in L, |s| \geq p$

$\exists x, y, z$ strings such that $s = xyz$, $|xy| \leq p$, and $|y| \neq 0$,

$\forall i \geq 0, \quad xy^iz \in L.$

"pf" L reg. Let N be an NFA recog L.
Let $p := \#$ states of N



② Can traverse the loop i times (any $i \geq 0$) ~~and~~ and still get the same accept state

$\therefore xy^iz \in L$.

$i \geq 2$ — "pumping up"

$i = 0$ — "pumping down"

What does it mean for L not pumpable?

L pumpable $\iff (\exists p \cdots)(\forall s \cdots)(\exists x,y,z \cdots)(\forall i \cdots)\underline{[xy^iz \in L]}$

L not pumpable $\iff \neg(\exists p \cdots)(\forall s \cdots)(\exists x,y,z)(\forall i \cdots)[xy^iz \in L]$

$\iff (\forall p \cdots)\neg[\qquad\qquad]$

$\iff (\forall p \cdots)(\exists s \cdots)\neg(\exists x,y,z \cdots)(\forall i)[xy^iz \in L]$

$\iff$ $\boxed{\begin{array}{l}(\forall p > 0)(\exists s \in L, |s| \geq p) \\ (\forall x,y,z \text{ such that } s = xyz, |xy| \leq p, |y| > 0) \\ (\exists i \geq 0)\underline{[xy^iz \notin L]}\end{array}}$

$\neg(\exists x) \cdots \iff (\forall x)\neg \cdots$

$\neg(\forall x) \cdots \iff (\exists x)\neg \cdots$

③ Ex: $\Sigma = \{0, 1\}$.

$L := \{0^n 1^n : n \geq 0\}$

Prop: L is not pumpable.

Proof: Given $p > 0$, let $s := 0^p 1^p$

[note: $s \in L$ and $|s| = 2p \geq p$]

Given x, y, z such that $s = xyz$,

$|xy| \leq p$, and $|y| > 0$

[By our choice of s, know that $y = 0^k$ (some $k > 0$)]

Let $i := 0$. (pump down)

Then $xy^0z = xz = 0^{p-k}1^p \notin L$ (fewer 0's than 1's) //

[Any choice of i work exept $i := 1$ NEVER works]

Ex: $L = \{0^m 1^n : m \geq n \geq 0\}$ is not pumpable.

Pf: Given $p > 0$, let $s := 0^p 1^{p-1}$ [$s \in L$, $|s| = 2p-1 \geq p$]

[Opponent: $x = 0^{p-1}$, $y = 0$, $z = 1^{p-1}$] doesn't work

Try again: Given $p > 0$, let $s := 0^p 1^p$.

4 Given x, y, z such that $s = xyz$, $|xy| \le p$, $|y| > 0$,

$[y = 0^k$ for some $k > 0]$

let $i := 0$

only value of i that works!

Then $xy^0z = xz = 0^{p-k}1^p \notin L$ because $p-k < p$.

$\therefore$ L is not pumpable. //

Ex: $\Sigma = \{a, b, c\}$

$L := \{w \in \Sigma^* ; |w|$ is odd and the middle symbol of w occurs nowhere else in $w\}$.

Prop: L is not pumpable.

Pf: Given $p > 0$, let $s := a^{p-1}bc^{p-1}$. ($s \in L$, $|s| = 2p-1 \ge p$)

Given x, y, z such that $s = xyz$, $|xy| \le p$, $|y| > 0$,

Let $i := 2$. $xyyz \notin L$ because:

y is either $\underbrace{a^k}_{\text{case 1}}$ (some $k > 0$) or $\underbrace{a^kb \ (k \ge 0)}_{\text{case 2}}$

In case 1, $xyyz = a^{p-1+k}bc^{p-1}$ either even length or middle symbol is a which occurs ≥ 2 times in $xyyz$

possible but messy

In case 2 similar //

(5) What about $s := a^p b c^p$

Given x, y, z, know that $y = a^k$ $(k > 0)$

So $xyyz = a^{p+k} b c^p$ either has even length ($\notin L$)

or odd length & middle symbol is a

(and there are $p+k \geq 2$ a's in $xyyz$). $\therefore \notin L$. //

Combine Pumping Lemma with closure properties to show non regularity.

Proof template to show L not regular:

"Assume L is regular. Then by such-and-such closure property, some other language L' is regular. But can show that L' is not pumpable, violating the Pumping Lemma ↯ $\therefore$ L is not reg."

Ex: $L = \{0^m 1^n : m \neq n\}$ $\quad \boxed{\Sigma = \{0,1\}}$

Proof: Suppose L is regular. Then

$\overline{L}$ is regular (reg langs closed under complement).

Therefore $\underline{\overline{L} \cap L(0^*1^*)}$ is regular (reg langs closed under intersection)

But $\overline{L} \cap L(0^*1^*) = \{0^m 1^n : \text{some } m, n \geq 0\} \cap \overline{L}$

$= \{0^m 1^n : m, n \geq 0 \text{ and } m = n\}$

(6) $= \{0^n 1^n : n \geq 0\}$

But this is not pumpable $\lightning$: Therefore L is not regular //

Pumping Lemma

Ex: $\Sigma = \{a, b, c\}$

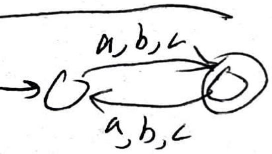
$L = \{w \in \Sigma^* :$ w has odd length and its middle symbol does occur elsewhere in $w\}$

Prop: L is not regular.

Pf: Suppose L is regular. Then $\overline{L}$ is regular.

Then $\underline{\overline{L} \cap \{w : |w| \text{ is odd}\}}$ is regular.

regular



But $\overline{L} \cap \{|w| \text{ is odd}\} = \{w \in \Sigma^* :$ w has odd length & middle symbol does not occur elsewhere$\}$ ← showed not pumpable

∴ not reg $\lightning$ //

Prop: $\{0^m 1^n : m \neq n\}$ is not pumpable.

Given $p > 0$, let $s := 0^p 1^{p+p!}$

Given x, y, z ... know that $y = 0^k$ $(1 \leq k \leq p)$

Then for any i, $xy^iz = 0^{p+(i-1)k} 1^{p+p!}$

~~the~~ Choose i so that $(i-1)k = p!$ i.e., $i := \frac{p!}{k} + 1$ //