① String homomorphisms (for closure properties) CSCE 355 2/16/22

Recall: Σ, Γ alphabets

A string homomorphism from Σ to Γ is a map $\varphi : \Sigma^* \to \Gamma^*$ such that $\forall x, y \in \Sigma^*$,

$$\underbrace{\varphi(xy)}_{\text{in } \Sigma^*} = \underbrace{\varphi(x)\varphi(y)}_{\Gamma^*}$$

Showed that $\varphi(\varepsilon) = \varepsilon$

Prop: φ is completely specified by what it does with strings of length 1 (symbols in Σ)

Proof sketch: $\forall w \in \Sigma^*$, write $w = w_1 w_2 \cdots w_n$ (each $w_i \in \Sigma$). Then

$$\varphi(w) = \varphi(w_1 \cdots w_n) = \varphi(w_1)\varphi(w_2 \cdots w_n)$$
$$= \varphi(w_1)\varphi(w_2)\varphi(w_3 \cdots w_n) = \cdots$$
$$= \varphi(w_1) \cdots \varphi(w_n) \quad \blacksquare$$

Converse: Any mapping $\Sigma \to \Gamma^*$ is uniquely extendable to a string homo. $\Sigma^* \to \Gamma^*$.

② Ex: $\Sigma = \{a,b,c\}$, $\Gamma = \{0,1\}$ φ str. homo Σ to Γ

$a \mapsto 01$
$b \mapsto 100$
$c \mapsto \varepsilon$

$\varphi(abcbabc) = 0110010001100$

Def: Σ, Γ alphabets, φ str. homo. Σ to Γ.
Let $L \subseteq \Sigma^*$. Define

$$\varphi(L) := \{\varphi(w) : w \in L\} \quad (\subseteq \Gamma^*)$$

Let $M \subseteq \Gamma^*$. Define

$$\varphi^{-1}(M) := \{w \in \Sigma^* : \varphi(w) \in M\}. \quad (\subseteq \Sigma^*)$$

$\varphi(L)$ is the image of L under φ
$\varphi^{-1}(M)$ " " inverse image of M under φ.

Thm: Let Σ, Γ, φ be as above. If $L \subseteq \Sigma^*$ is regular, then $\varphi(L)$ is regular.

Proof: We transform any regex r over Σ into a regex $\varphi(r)$ over Γ such that

$$L(\varphi(r)) = \varphi(L(r))$$

(3) This suffices: any reg lang over Σ is of the form $L(r)$ for some regex r over Σ.
Then $\varphi(L(r)) = L(\underbrace{\varphi(r)}_{\text{regex over } \Gamma})$ is regular.

We define $\varphi(r)$ according to these rules:

	r	$\varphi(r)$	
	$\emptyset$	$\emptyset$	
$(a \in \Sigma)$	a	$\varphi(a)$	string in Γ^*, interpreted as a regex over Γ
s, t regexes over Σ	$s+t$	$\varphi(s)+\varphi(t)$	
	st	$\varphi(s)\varphi(t)$	because φ is a string homo.
	s^*	$\varphi(s)^*$	

EX: φ: $a \longmapsto 01$, $b \longmapsto 100$, $c \longmapsto \varepsilon$

$$\varphi\left(((ab)^* + (a+bbca)^*)^*\right) = \ldots =$$
$$= ((01100)^* + (01 + 10010001)^*)^*$$

4. Thm: Let Σ, Γ, φ be as above. $M \subseteq \Gamma^*$. If M is regular, then $\varphi^{-1}(M)$ is regular.

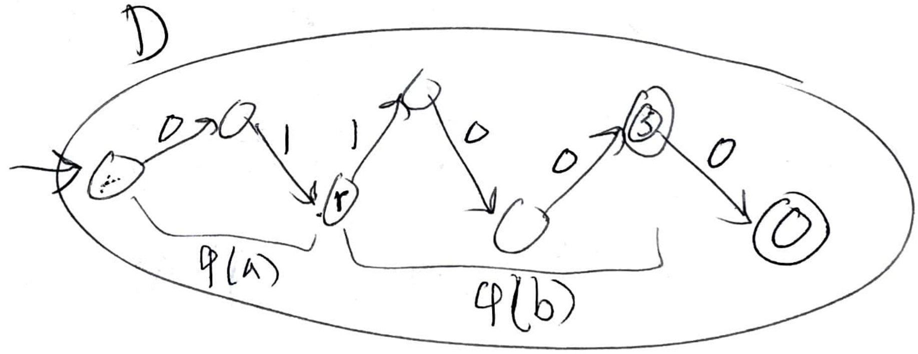
Proof sketch: Let $D = \langle Q, \Gamma, \delta, q_0, F \rangle$ be a DFA recognizing M.

We construct a DFA

$$D' := \langle Q, \underline{\Sigma}, \underline{\delta'}, q_0, F \rangle.$$

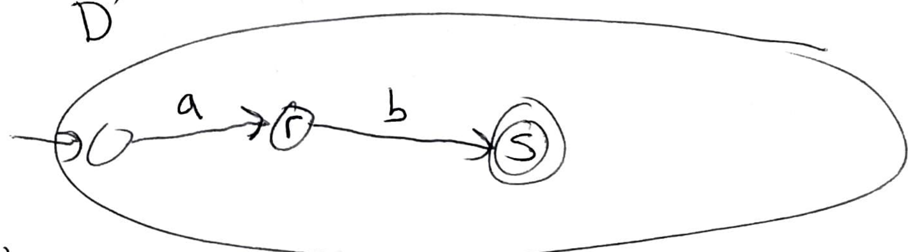
Idea:

D



$$\varphi : \begin{array}{l} a \mapsto 01 \\ b \mapsto 100 \\ c \mapsto \varepsilon \end{array}$$

D'



D' accepts ab iff D accepts $\varphi(ab)$, iff $\varphi(ab) \in M$, iff $ab \in \varphi^{-1}(M)$.

5 Formal construction: For every $a \in \Sigma'$ and state $q \in Q$, define

$$\delta'(q,a) := \hat{\delta}(q, \varphi(a)).$$

Follows that for any string $w \in \Sigma'^*$

$$\hat{\delta'}(q_0, w) = \hat{\delta}(q_0, \varphi(w))$$

Thus

D' accepts w iff $\hat{\delta'}(q_0, w) \in F$

iff $\hat{\delta}(q_0, \varphi(w)) \in F$ iff D accepts $\varphi(w)$

$\therefore L(D') = \varphi^{-1}(L(D)) = \varphi^{-1}(M)$. $\blacksquare$

Ex: $L \subseteq \Sigma'^*$ $(\Sigma' = \{a, b, c\})$.

Let BbeforeA(L) be the set of all strings obtained from strings in L by inserting a "b" right before every occurrence of "a" in the string

E.g., BbeforeA({cabac}) = {cbabbac}

⑥ Show that if L is regular, then BbeforeA(L) is regular.

Pf: Let φ be the str homo Σ to Σ such that

$$a \xmapsto{\varphi} ba$$
$$b \xmapsto{\varphi} b$$
$$c \xmapsto{\varphi} c$$

Note that BbeforeA$(L) = \varphi(L)$, hence regular if L is regular (by the 1st Thm).

" Remove all a's;

$$a \longmapsto \varepsilon$$
$$b \longmapsto b$$
$$c \longmapsto c$$

<u>Proving a language is not regular</u>

Main tool: <u>Pumping Lemma (for regular langs.)</u>

⑦ <u>Def</u>: Σ alphabet, $L \subseteq \Sigma^*$ any lang. over Σ.

Say that L is <u>pumpable</u> if

There exists an integer $p > 0$ (the "pumping length")
such that

For every string $s \in L$ where $|s| \geq p$,

There exist strings $x, y, z \in \Sigma^*$
such that
1) $s = xyz$
2) $|xy| \leq p$
3) $|y| > 0$ (i.e. $y \neq \varepsilon$)

and ..

For every integer $i \geq 0$,

$$xy^iz \in L.$$

$i = 0$: $xy^0z = xz$ "pumping down"

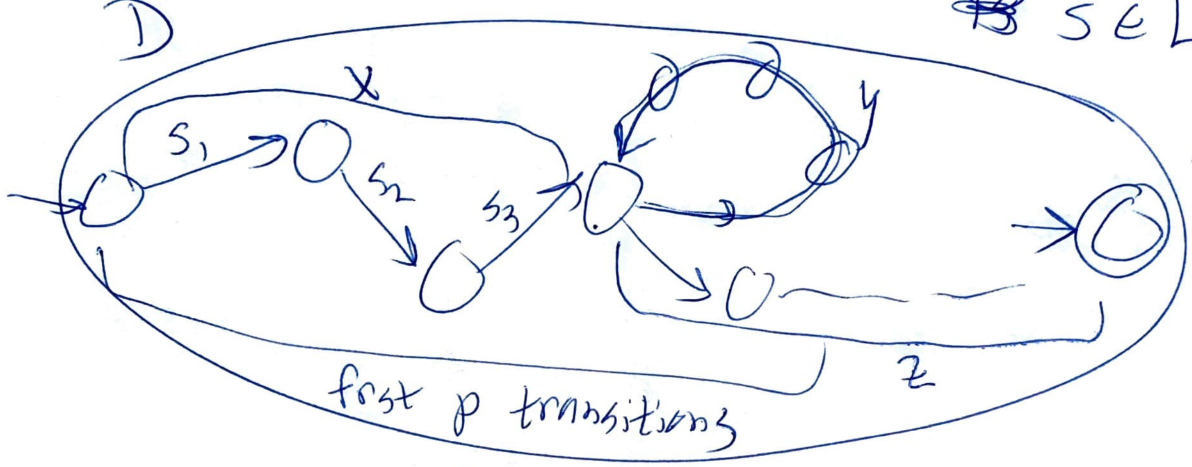
$i \geq 2$: $xy^iz = xyy\cdots z$ "pumping up"

<u>Pumping Lemma</u>: Every regular language is pumpable.

8 **Proof idea**: Let ~~B~~ L be regular, D a DFA recognizing L.

Let p be the number of D's states

~~B~~ $s \in L$, $|s| \geq p$



must repeat some state.

$s = xyz$, $|xy| \leq p$, $|y| > 0$