CSCE 355
1/19/2022

String induction

Basic idea:

Let w be any string over alphabet Σ.
Exactly one of the following ~~ab~~ holds:

base case → 1. $w = \varepsilon$ $(|w| = |\varepsilon| = 0)$

inductive case → 2. There exist unique $x \in \Sigma^*$ and $a \in \Sigma$ such that

$$w = xa$$

a is the last symbol of w
x " " principal prefix of w
$(|w| = |x| + 1)$

Theorem: Let $w \in \Sigma^*$ be any string. Exactly one of the following holds:

1. $w = \varepsilon$ $(|w| = 0)$
2. There exist ~~str~~ $x \in \Sigma^*$ and $a \in \Sigma$ such that $w = ax$ $(|w| = |x| + 1)$

(2) **Proof**: Base case: $w = \varepsilon$. Then $|w| = 0$

(2 can't happen, because if it did, then $|w| = |x| + 1 > 0$ ↯)

Inductive case: $w \neq \varepsilon$. By the basic principle, there exist $x \in \Sigma^*$ and $b \in \Sigma$ such that $w = xb$.

$|x| < |w|$, so we can assume (inductive hypothesis) that x satisfies the thm, i.e., either $x = \varepsilon$ or there exist $y \in \Sigma^*$ and $a \in \Sigma$ such that $x = ay$

But then, $w = xb = (ay)b = a(yb)$
$= az$, where $z = yb \in \Sigma^*$.

So $a \in \Sigma$, $z \in \Sigma^*$ such that $w = az$

$\therefore$ Theorem holds for w.

$\therefore$ Thm holds by induction on $|w|$. ∎

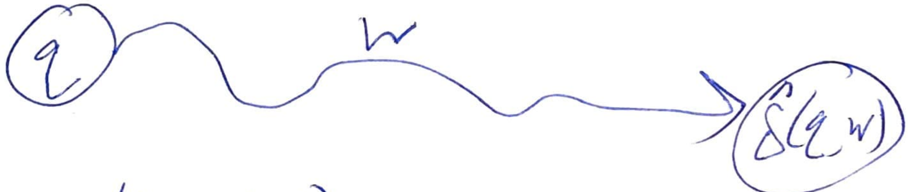
(3) Recall: $A = \langle Q, \Sigma, \delta, q_0, F \rangle$

a DFA. We define the extended transition function

$$\hat{\delta} : Q \times \Sigma^* \longrightarrow Q$$

such that

$\hat{\delta}(q, w)$ is the final state after reading string w starting in state q.



A accepts w (by def) iff

$$\hat{\delta}(q_0, w) \in F.$$

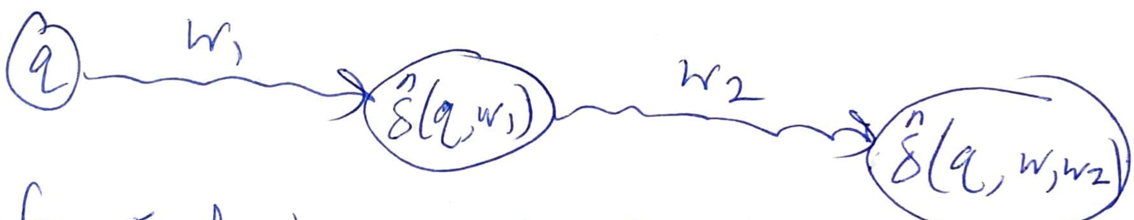
Inductive definition of $\hat{\delta}$, based on $|w|$.

1. (Base case): $\forall q \in Q$, $\hat{\delta}(q, \varepsilon) := q$
2. (Inductive case): $w = xa$ (unique $x \in \Sigma^*$ and $a \in \Sigma$).

(4) $\forall q \in Q$, $\hat{\delta}(q, w) \stackrel{?}{=} \delta(\underline{\hat{\delta}(q, x)}, a)$

<u>Prop</u>: For any strings $w_1, w_2 \in \Sigma^*$ and any state $q \in Q$,

$$\hat{\delta}(q, w_1 w_2) = \hat{\delta}(\underline{\hat{\delta}(q, w_1)}, w_2)$$



<u>Proof</u>: <u>Induction on $|w_2|$</u> of the statement:

$$\boxed{\forall q \in Q, \forall w_1 \in \Sigma^*, \hat{\delta}(q, w_1 w_2) = \hat{\delta}(\hat{\delta}(q, w_1), w_2)}$$

Case 1. $w_2 = \varepsilon$. Then $w_1 w_2 = w_1$. For any $q \in Q$ any $w_1 \in \Sigma^*$,

$$\underline{\hat{\delta}(q, w_1 w_2)} = \boxed{\hat{\delta}(q, w_1)} = \hat{\delta}(\boxed{\hat{\delta}(q, w_1)}, \varepsilon)$$

base case of def of $\hat{\delta}$.

$$\cancel{\hat{\delta}(q, w_1)), w_2)} \qquad [\varepsilon = w_2]$$

$$= \hat{\delta}(\hat{\delta}(q, w_1), w_2)$$

Case 2. $w_2 \neq \varepsilon$. So $w_2 = xa$ for unique $x \in \Sigma^*$ and $a \in \Sigma$.

By inductive hypothesis, $\forall q \in Q$, $\forall w_1 \in \Sigma^*$

$$\hat{\delta}(q, w_1 x) = \hat{\delta}(\underline{\hat{\delta}(q, w_1)}, x)$$

Given any $q \in Q$ and $w_1 \in \Sigma^*$, let $r := \hat{\delta}(q, w_1)$

WTS (want to show) $\hat{\delta}(q, w_1 w_2) = \hat{\delta}(r, w_2)$.

$$\hat{\delta}(q, w_1 w_2) = \hat{\delta}(q, w_1 xa) \qquad [w_2 = xa]$$

$$= \delta(\underline{\hat{\delta}(q, w_1 x)}, a) \qquad [\text{inductive case of def of } \hat{\delta}]$$

$$= \delta(\hat{\delta}(r, x), a) \qquad [\text{inductive hypothesis}]$$

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$$\cancel{= \delta(\hat{\delta}(\hat{\delta}(q, w_1), x), a)}$$

$$= \hat{\delta}(r, xa) \qquad [\text{ind part of def of } \hat{\delta}]$$

$$= \hat{\delta}(r, w_2) \qquad [w_2 = xa]$$

Q.E.D.

∴ By induction statement holds for all w_2, i.e.,

$$\forall w_2 \in \Sigma^* \ \forall q \in Q, \forall w_1 \in \Sigma^*$$
$$\hat{\delta}(q, w_1 w_2) = \hat{\delta}(\hat{\delta}(q, w_1), w_2)$$

$\blacksquare$

Can prove concat is associative, i.e.,

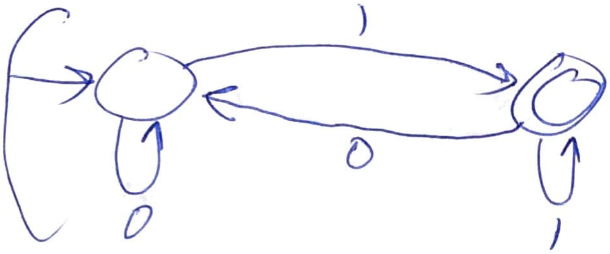
$x(yz) = (xy)z$ by induction on $|z|$.

Ex: $\Sigma = \{0, 1\}$

$L = \{ x \in \Sigma^* :$ the 2nd to last symbol of x is $1\}$

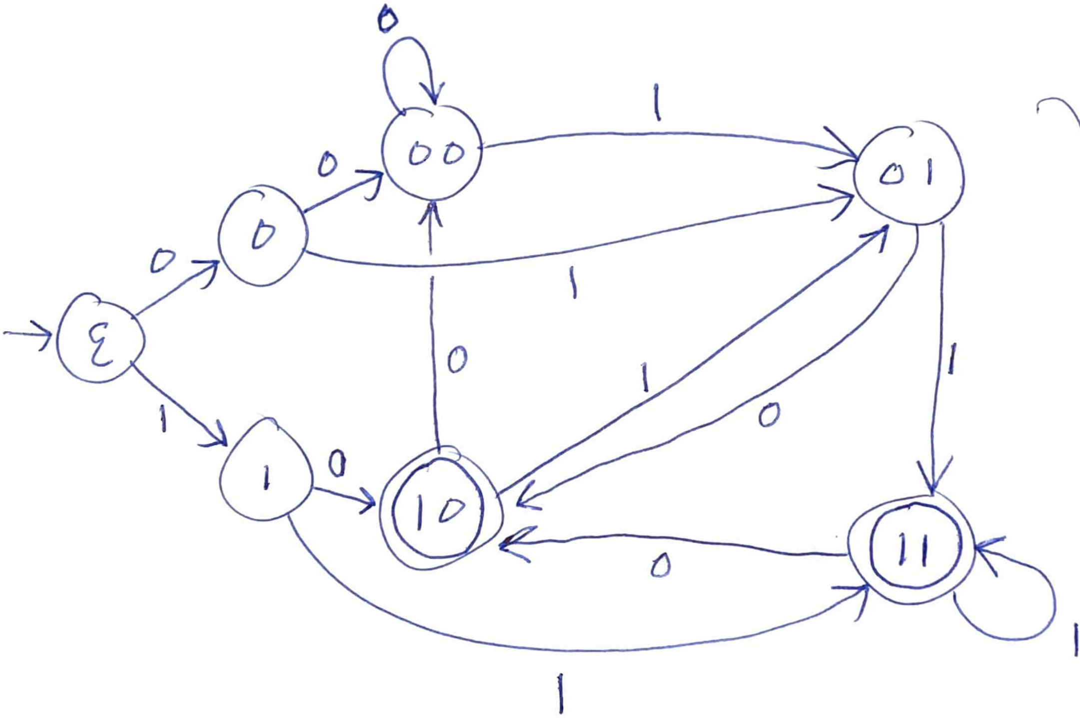
(7) Recall: $L' = \{x \in \Sigma^* : \text{the last symbol of } x \text{ is } 1\}$

2 state



recognizes L'.

$x = 0010110$



7 states

(4 states in necessary & sufficient)

NFA (nondeterministic finite automaton) has no restriction on the number of arrows leaving a state with a given label in Σ. :

8 NFA for L: