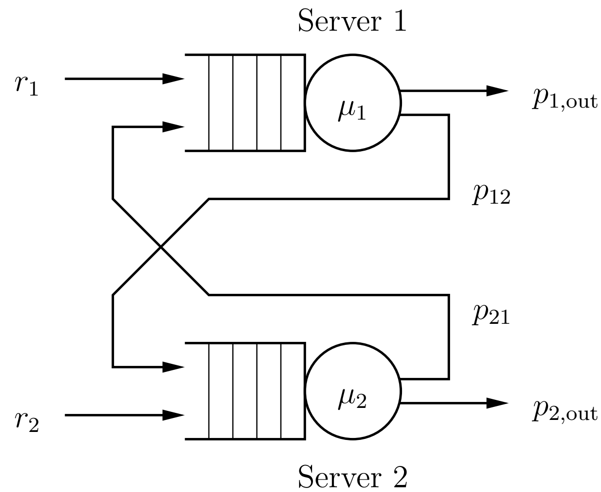
CSCE 317
Spring 2014
First Midterm Exam, Answer Key

1. **(Probabilistic routing between servers; 15 points total)** Consider the network below, consisting of two servers:



Jobs leaving each server are routed at random according to the probabilities shown, where $p_{1,\text{out}} = 0.6$ and $p_{2,\text{out}} = 0.1$. Incoming jobs from the outside are sent to the two servers at average rates r_1 and r_2, respectively. Suppose that $r_2 = 5$ (jobs/sec).

(a) (10 points) For $i = 1, 2$, find an exact, simplified expression (in terms of r_1) for the average rate λ_i of jobs entering Server i.

(b) (5 points) Suppose that $\mu_1 = \mu_2 = 10$. How big can r_1 be before the system becomes unstable? Which server causes the bottleneck?

Answer:

(a) We deduce that $p_{12} = 1 - p_{1,\text{out}} = 0.4$ and $p_{21} = 1 - p_{2,\text{out}} = 0.9$ to get the two equations

$$\lambda_1 = r_1 + p_{21}\lambda_2 = r_1 + 0.9\lambda_2 \,,$$
$$\lambda_2 = r_2 + p_{12}\lambda_1 = 5 + 0.4\lambda_1 \,.$$

Substituting, we get

$$\begin{aligned}
\lambda_1 &= r_1 + 0.9(5 + 0.4\lambda_1) = r_1 + 4.5 + 0.36\lambda_1 \\
0.64\lambda_1 &= r_1 + 4.5 \\
\lambda_1 &= \frac{r_1 + 4.5}{0.64} = \frac{r_1 + 9/2}{16/25} = \frac{25}{16}r_1 + \frac{225}{32} = 1.5625r_1 + 7.03125 \,, \\
\lambda_2 &= 5 + \frac{2}{5}\left(\frac{25}{16}r_1 + \frac{225}{32}\right) = \frac{5}{8}r_1 + \frac{125}{16} = 0.625r_1 + 7.8125 \,.
\end{aligned}$$

(b) Setting $\lambda_1 < 10$ and solving for r_1 gives $r_1 < 1.9$. Setting $\lambda_2 < 10$ and solving for r_1 gives $r_1 < 3.5$. Since both must be true, we have that $r_1 < 1.9$, and Server 1 is the bottleneck.

2. **(Bernoulli trials; 10 points)** At an arcade game, you get to toss a ball at a target six times. To win the prize, you must hit the target at least twice out of the six tries. Your accuracy is such that each toss has a 0.1 chance of hitting the target, independent of the other tosses. What is the probability that you win the prize?

   Answer: Let N be the number of times you hit the target. Then $N \sim \text{Binomial}(6, 0.1)$, and so

$$\begin{aligned}\Pr[N \geq 2] &= 1 - \Pr[N < 2] = 1 - (\Pr[N = 0] + \Pr[N = 1]) \\ &= 1 - \binom{6}{0} 0.1^0 0.9^6 - \binom{6}{1} 0.1^1 0.9^5 = 1 - 0.9^6 - (6)(0.1)(0.9^5) \\ &= 1 - 0.9^5(0.9 + 0.6) = 1 - (1.5)0.9^5 = 0.114265\ .\end{aligned}$$

   Since I promised that a calculator would not be needed, I'll accept any reasonably simple expression short of the actual decimal value.

3. **(Bayesian inference; 10 points)** A test for a particular genetic marker is 90% accurate. That is, 90% of those with the genetic marker test positive, and the rest (falsely) test negative. Similarly, 90% of those without the genetic marker test negative, and the rest (falsely) test positive. The genetic marker is known to exist in 5% of the population at large. A person selected at random is given the test and tests positive. What is the probability that that person has the genetic marker?

   Answer: Let P be the event that the person tests positive, and let M be the event that the person has the genetic marker. We want $\Pr[M \mid P]$, and we know that $\Pr[P \mid M] = 0.9$, that $\Pr[P \mid \overline{M}] = 0.1$, and that $\Pr[M] = 0.05$. Thus

$$\Pr[M \mid P] = \frac{\Pr[P \mid M]\Pr[M]}{\Pr[P \mid M]\Pr[M] + \Pr[P \mid \overline{M}]\Pr[\overline{M}]} = \frac{(0.9)(0.05)}{(0.9)(0.05) + (0.1)(0.95)} = \frac{9}{28} \approx 0.321\ .$$

4. **(A continuous random variable; 5 points)** Sarah's new disk runs continuously, and the time T until the next failure is exponentially distributed with mean 10 years. Starting exactly one year from now, Sarah will be overseas in the Army for exactly two years. What is the probability that her disk will fail for the first time while she is overseas?

   Answer: Measuring time in years, we have $T \sim \text{Exp}(0.1)$, and we want to find $\Pr[1 \leq T \leq 3]$. We have

$$\Pr[1 \leq T \leq 2] = F_T(3) - F_T(1) = (1 - e^{-0.3}) - (1 - e^{-0.1}) = e^{-0.1} - e^{-0.3} = \frac{e^{0.2} - 1}{e^{0.3}} \approx 0.164\ .$$

5. **(Joint distributions; 10 points)** Alice chooses a uniformly random real number A between 1 and 3, and Bob (independently) chooses a uniformly random real number B between 2 and 5. (That is, $A \sim \text{Uniform}(1, 3)$ and $B \sim \text{Uniform}(2, 5)$ and $A \perp B$.) What is the probability that Alice's number is greater than Bob's? [You may use any method you like, but you must show your work.]

Answer: $\Pr[A > B] = 1/12$ (one-twelfth). We can get this in two ways: following the book's method; or just applying symmetry, using the fact that the distributions are both uniform.

By the book, using the fact that $A \perp B$:

$$\Pr[A > B] = \iint_{x>y} f_{A,B}(x,y)\,dx\,dy = \iint_{x>y} f_A(x) f_B(y)\,dx\,dy\ ,$$

where $f_A(x) = 1/2$ when $1 \le x \le 3$ and is 0 otherwise, and $f_B(y) = 1/3$ when $2 \le y \le 5$ and is 0 otherwise. We then have

$$\begin{aligned}
\iint_{x>y} f_A(x) f_B(y)\,dx\,dy &= \frac{1}{3}\int_{y=2}^{5}\int_{x=y}^{\infty} f_A(x)\,dx\,dy \\
&= \frac{1}{6}\int_{y=2}^{3}\int_{x=y}^{3} dx\,dy \qquad\qquad \text{(because } f_A(x) = 0 \text{ when } x > 3\text{)} \\
&= \frac{1}{6}\int_{y=2}^{3} (3-y)\,dy = \frac{1}{12}\ .
\end{aligned}$$

Alternatively, we notice that the only way to have $A > B$ is if $A > 2$ and $B < 3$. So

$$\begin{aligned}
\Pr[A > B] &= \Pr[A > B \;\&\; A > 2 \;\&\; B < 3] \\
&= \Pr[A > B \mid A > 2 \;\&\; B < 3]\,\Pr[A > 2 \;\&\; B < 3] \\
&= \Pr[A > B \mid A > 2 \;\&\; B < 3]\,\Pr[A > 2]\,\Pr[B < 3] \qquad \text{(because } A \perp B\text{)} \\
&= \Pr[A > B \mid A > 2 \;\&\; B < 3] \cdot \frac{1}{2} \cdot \frac{1}{3} \\
&= \frac{1}{6}\Pr[A > B \mid A > 2 \;\&\; B < 3]\ .
\end{aligned}$$

Now, on the condition that A and B are both uniformly distributed between 2 and 3, by symmetry, we must have $\Pr[A > B] = \Pr[A < B]$, because there is no bias towards A or B being greater. Since $\Pr[A > B] + \Pr[A = B] + \Pr[A < B] = 1$ and $\Pr[A = B] = 0$, we get $\Pr[A > B] = 1/2$. Putting the condition back explicitly, $\Pr[A > B \mid A > 2 \;\&\; B < 3] = 1/2$, and so unconditionally,

$$\Pr[A > B] = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}\ .$$