5 Parallel Algorithms (continued)

Definition 5.1. *The* **diameter** *of a network is the maximum distance between any two nodes.*

Definition 5.2. *The* **bisection width** *of a network is the minimum number of arcs that one must cut in order to separate the network into two halves ("half" to within one node).*

In general we will want to compare and measure network communication capability based on four criteria.

1. diameter
2. bisection width
3. number of edges per node
4. the maximum edge length needed to realize the network

Definition 5.3. *A* **hypercube** *is a network with* n *nodes, numbered* $0, 1, ..., n-1$*, in which nodes are connected if they differ by* 1 *in the Hamming distance between the binary representations of their node numbers.*

Definition 5.4. *A* **cube connected cycles** *network is a* k*-dimensional hypercube in which the nodes of the hypercube have been replaced by cycles of* k *nodes with each node connecting along one edge of the hypercube and to two nodes of the* k*-cycle.*

Definition 5.5. *A* **shuffle exchange** *network consists of* $n = 2^k$ *nodes with two kinds of connections. The* **exchange** *connections are bidirectional links between pairs of nodes that differ in the least significant bit. The* **shuffle** *connections are unidirectional links from node* k *and node* $2k \pmod{n-1}$.

Definition 5.6. *An* **omega network** *is a composition of* k *shuffle exchange networks.*

type	nodes	diameter	bisection width	# edges ? constant?	constant edge length?
1-D mesh	k	$k-1$	1	Y	Y
2-D mesh	k^2	$2(k-1)$	k	Y	Y
3-D mesh	k^3	$3(k-1)$	k^2	Y	Y
binary tree	2^k-1	$2(k-1)$	1	Y	N
butterfly	$(k+1)2^k$	$2k$	2^k	Y	N
hypercube	2^k	k	2^{k-1}	N	N
cube conn cycles	$k2^k$	$2k$	2^{k-1}	Y	N
shuffle exchange	2^k	$2k-1$	$\geq 2^{k-1}/k$	Y	N

Theorem 5.7. *A dilation-*1 *embedding of a complete binary tree of height* n *into a hypercube of dimension* $n+1$ *does not exist for* $n > 1$.

Proof. Note that a complete binary tree of height n has $2^{n+1}-1$ nodes, and an $n+1$-dimensional hypercube has 2^{n+1} nodes. So the optimal embedding, if it were to exist, would be of a height-n tree into an $n+1$ dimensional cube. And the optimal embedding would have to use all the nodes of the hypercube except one.

Now, a tree of height n for n odd has more than half its nodes at an odd distance from the root. (height n, $2^{n+1} - 1$ nodes in all, and 2^n nodes that are the leaves at height exactly n). It is therefore not possible to embed the tree into the cube.

If, on the other hand, the height is even, then more than half the nodes are an even distance away from the root. Mutatis mutandis, the embedding isn't possible. □

We can't do as well as we would like to. However, we can do almost as well.

Theorem 5.8. *A dilation-*2 *embedding of a complete binary tree of height n into a hypercube of dimension $n + 1$ exists for all $n > 1$.*

Theorem 5.9. *A dilation-*1 *embedding of a balanced binary tree of height n into a hypercube of dimension $n + 2$ exists for all $n > 1$.*

In a different direction, we cannot do as well as we would like for a mesh, but once again we can do the next best thing.

Theorem 5.10. *Any* 2*-dimensional mesh with n vertices can be embedded into a $\lceil \lg n \rceil$-dimensional hypercube with dilation 2.*