

## 8.4 Polynomial Multiplication by Fourier Transforms

### 8.5 Example

We're going to multiply

$$f(x) = 1 + 2x + 3x^2$$

and

$$g(x) = 9 + 8x + 7x^2 + 6x^3$$

using a Fourier Transform.

First of all, we'll do the multiplication the long way so we'll have the answer.

$$\begin{array}{r}
 9 \ 18 \ 27 \ 0 \ 0 \ 0 \\
 + \ 0 \ 8 \ 16 \ 24 \ 0 \ 0 \\
 + \ 0 \ 0 \ 7 \ 14 \ 21 \ 0 \\
 + \ 0 \ 0 \ 0 \ 6 \ 12 \ 18 \\
 \hline
 9 \ 26 \ 50 \ 44 \ 33 \ 18
 \end{array}$$

or

$$h(x) = 9 + 26x + 50x^2 + 44x^3 + 33x^4 + 18x^5$$

Now, in order to do this as a power-of-two transform, we'll have to do an eight point transform, since 8 is the least power of 2 larger than the degree (5) of the product.

We do the multiplication

$$\begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the multiplication

$$\begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 8 \\ 7 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

to get two column vectors

$$\begin{pmatrix} 1 \cdot \omega^0 + 2 \cdot \omega^0 + 3 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 \\ 1 \cdot \omega^0 + 2 \cdot \omega^1 + 3 \cdot \omega^2 + 0 \cdot \omega^3 + 0 \cdot \omega^4 + 0 \cdot \omega^5 + 0 \cdot \omega^6 + 0 \cdot \omega^7 \\ 1 \cdot \omega^0 + 2 \cdot \omega^2 + 3 \cdot \omega^4 + 0 \cdot \omega^6 + 0 \cdot \omega^0 + 0 \cdot \omega^2 + 0 \cdot \omega^4 + 0 \cdot \omega^6 \\ 1 \cdot \omega^0 + 2 \cdot \omega^3 + 3 \cdot \omega^6 + 0 \cdot \omega^1 + 0 \cdot \omega^4 + 0 \cdot \omega^7 + 0 \cdot \omega^2 + 0 \cdot \omega^5 \\ 1 \cdot \omega^0 + 2 \cdot \omega^4 + 3 \cdot \omega^0 + 0 \cdot \omega^4 + 0 \cdot \omega^0 + 0 \cdot \omega^4 + 0 \cdot \omega^0 + 0 \cdot \omega^4 \\ 1 \cdot \omega^0 + 2 \cdot \omega^5 + 3 \cdot \omega^2 + 0 \cdot \omega^7 + 0 \cdot \omega^4 + 0 \cdot \omega^1 + 0 \cdot \omega^6 + 0 \cdot \omega^3 \\ 1 \cdot \omega^0 + 2 \cdot \omega^6 + 3 \cdot \omega^4 + 0 \cdot \omega^2 + 0 \cdot \omega^0 + 0 \cdot \omega^6 + 0 \cdot \omega^4 + 0 \cdot \omega^2 \\ 1 \cdot \omega^0 + 2 \cdot \omega^7 + 3 \cdot \omega^6 + 0 \cdot \omega^5 + 0 \cdot \omega^4 + 0 \cdot \omega^3 + 0 \cdot \omega^2 + 0 \cdot \omega^1 \end{pmatrix}$$

and

$$\begin{pmatrix} 9 \cdot \omega^0 + 8 \cdot \omega^0 + 7 \cdot \omega^0 + 6 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 \\ 9 \cdot \omega^0 + 8 \cdot \omega^1 + 7 \cdot \omega^2 + 6 \cdot \omega^3 + 0 \cdot \omega^4 + 0 \cdot \omega^5 + 0 \cdot \omega^6 + 0 \cdot \omega^7 \\ 9 \cdot \omega^0 + 8 \cdot \omega^2 + 7 \cdot \omega^4 + 6 \cdot \omega^6 + 0 \cdot \omega^0 + 0 \cdot \omega^2 + 0 \cdot \omega^4 + 0 \cdot \omega^6 \\ 9 \cdot \omega^0 + 8 \cdot \omega^3 + 7 \cdot \omega^6 + 6 \cdot \omega^1 + 0 \cdot \omega^4 + 0 \cdot \omega^7 + 0 \cdot \omega^2 + 0 \cdot \omega^5 \\ 9 \cdot \omega^0 + 8 \cdot \omega^4 + 7 \cdot \omega^0 + 6 \cdot \omega^4 + 0 \cdot \omega^0 + 0 \cdot \omega^4 + 0 \cdot \omega^0 + 0 \cdot \omega^4 \\ 9 \cdot \omega^0 + 8 \cdot \omega^5 + 7 \cdot \omega^2 + 6 \cdot \omega^7 + 0 \cdot \omega^4 + 0 \cdot \omega^1 + 0 \cdot \omega^6 + 0 \cdot \omega^3 \\ 9 \cdot \omega^0 + 8 \cdot \omega^6 + 7 \cdot \omega^4 + 6 \cdot \omega^2 + 0 \cdot \omega^0 + 0 \cdot \omega^6 + 0 \cdot \omega^4 + 0 \cdot \omega^2 \\ 9 \cdot \omega^0 + 8 \cdot \omega^7 + 7 \cdot \omega^6 + 6 \cdot \omega^5 + 0 \cdot \omega^4 + 0 \cdot \omega^3 + 0 \cdot \omega^2 + 0 \cdot \omega^1 \end{pmatrix}$$

where we have left in all the coefficients because we're going to make them all collapse later.

We do the componentwise multiplication and we know what we're going to get because this problem is small enough that we can do the multiplication

by hand.

$$hvec = \begin{pmatrix} 9 \cdot \omega^0 + 26 \cdot \omega^0 + 50 \cdot \omega^0 + 44 \cdot \omega^0 + 33 \cdot \omega^0 + 18 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 \\ 9 \cdot \omega^0 + 26 \cdot \omega^1 + 50 \cdot \omega^2 + 44 \cdot \omega^3 + 33 \cdot \omega^4 + 18 \cdot \omega^5 + 0 \cdot \omega^6 + 0 \cdot \omega^7 \\ 9 \cdot \omega^0 + 26 \cdot \omega^2 + 50 \cdot \omega^4 + 44 \cdot \omega^6 + 33 \cdot \omega^0 + 18 \cdot \omega^2 + 0 \cdot \omega^4 + 0 \cdot \omega^6 \\ 9 \cdot \omega^0 + 26 \cdot \omega^3 + 50 \cdot \omega^6 + 44 \cdot \omega^1 + 33 \cdot \omega^4 + 18 \cdot \omega^7 + 0 \cdot \omega^2 + 0 \cdot \omega^5 \\ 9 \cdot \omega^0 + 26 \cdot \omega^4 + 50 \cdot \omega^0 + 44 \cdot \omega^4 + 33 \cdot \omega^0 + 18 \cdot \omega^4 + 0 \cdot \omega^0 + 0 \cdot \omega^4 \\ 9 \cdot \omega^0 + 26 \cdot \omega^5 + 50 \cdot \omega^2 + 44 \cdot \omega^7 + 33 \cdot \omega^4 + 18 \cdot \omega^1 + 0 \cdot \omega^6 + 0 \cdot \omega^3 \\ 9 \cdot \omega^0 + 26 \cdot \omega^6 + 50 \cdot \omega^4 + 44 \cdot \omega^2 + 33 \cdot \omega^0 + 18 \cdot \omega^6 + 0 \cdot \omega^4 + 0 \cdot \omega^2 \\ 9 \cdot \omega^0 + 26 \cdot \omega^7 + 50 \cdot \omega^6 + 44 \cdot \omega^5 + 33 \cdot \omega^4 + 18 \cdot \omega^3 + 0 \cdot \omega^2 + 0 \cdot \omega^1 \end{pmatrix}$$

Now we multiply this column vector by the inverse matrix for the Fourier Transform. This matrix is

$$F^{-1} = (\omega^{-ij}) = \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \end{pmatrix}$$

So let's do the multiplication line by line. The first line of the result will be the dot product of the first line of  $F^{-1}$  with  $hvec$ , that is, of

$$(\omega^0 \ \omega^0 \ \omega^0 \ \omega^0 \ \omega^0 \ \omega^0 \ \omega^0 \ \omega^0)$$

with *hvec*. What we get is just the sum of the lines of *hvec*, since  $\omega^0 = 1$ :  
This is

$$\begin{aligned}
& 9 \cdot \omega^0 + 26 \cdot \omega^0 + 50 \cdot \omega^0 + 44 \cdot \omega^0 + 33 \cdot \omega^0 + 18 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 \\
& \quad + \\
& 9 \cdot \omega^0 + 26 \cdot \omega^1 + 50 \cdot \omega^2 + 44 \cdot \omega^3 + 33 \cdot \omega^4 + 18 \cdot \omega^5 + 0 \cdot \omega^6 + 0 \cdot \omega^7 \\
& \quad + \\
& 9 \cdot \omega^0 + 26 \cdot \omega^2 + 50 \cdot \omega^4 + 44 \cdot \omega^6 + 33 \cdot \omega^0 + 18 \cdot \omega^2 + 0 \cdot \omega^4 + 0 \cdot \omega^6 \\
& \quad + \\
& 9 \cdot \omega^0 + 26 \cdot \omega^3 + 50 \cdot \omega^6 + 44 \cdot \omega^1 + 33 \cdot \omega^4 + 18 \cdot \omega^7 + 0 \cdot \omega^2 + 0 \cdot \omega^5 \\
& \quad + \\
& 9 \cdot \omega^0 + 26 \cdot \omega^4 + 50 \cdot \omega^0 + 44 \cdot \omega^4 + 33 \cdot \omega^0 + 18 \cdot \omega^4 + 0 \cdot \omega^0 + 0 \cdot \omega^4 \\
& \quad + \\
& 9 \cdot \omega^0 + 26 \cdot \omega^5 + 50 \cdot \omega^2 + 44 \cdot \omega^7 + 33 \cdot \omega^4 + 18 \cdot \omega^1 + 0 \cdot \omega^6 + 0 \cdot \omega^3 \\
& \quad + \\
& 9 \cdot \omega^0 + 26 \cdot \omega^6 + 50 \cdot \omega^4 + 44 \cdot \omega^2 + 33 \cdot \omega^0 + 18 \cdot \omega^6 + 0 \cdot \omega^4 + 0 \cdot \omega^2 \\
& \quad + \\
& 9 \cdot \omega^0 + 26 \cdot \omega^7 + 50 \cdot \omega^6 + 44 \cdot \omega^5 + 33 \cdot \omega^4 + 18 \cdot \omega^3 + 0 \cdot \omega^2 + 0 \cdot \omega^1
\end{aligned}$$

and now we can see the advantage of not collapsing things earlier. Adding up the first column of the above tableau we get

$$9 \cdot \omega^0 = 9 \cdot 1 = 9$$

added up 8 times, which is 72. In every other column we get the coefficient

times

$$\omega^0 + \omega^1 + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7$$

But this sum is zero, because these are roots of unity, so we have

$$0 = \omega^8 - 1 = (\omega^1 - 1) \cdot (\omega^0 + \omega^1 + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7)$$

The product is zero, but since the first factor  $(\omega^1 - 1)$  isn't zero, the second factor must be zero.

So, going down every column except the column for the coefficient 9, we get a coefficient (26, 50, 44, 33, 18) times zero. The first line of the product of  $F^{-1}$  and *hvec* is thus just the 72.

Now we'll do the second line, and this will be all we'll need to see the pattern. Take the second line of  $F^{-1}$ ,

$$\left( \omega^0 \ \omega^7 \ \omega^6 \ \omega^5 \ \omega^4 \ \omega^3 \ \omega^2 \ \omega^1 \right),$$

and multiply this times  $hvec$ . We get

$$\begin{aligned}
& 9 \cdot \omega^0 + 26 \cdot \omega^0 + 50 \cdot \omega^0 + 44 \cdot \omega^0 + 33 \cdot \omega^0 + 18 \cdot \omega^0 + 0 \cdot \omega^0 + 0 \cdot \omega^0 \\
& \quad + \\
& 9 \cdot \omega^7 + 26 \cdot \omega^0 + 50 \cdot \omega^1 + 44 \cdot \omega^2 + 33 \cdot \omega^3 + 18 \cdot \omega^4 + 0 \cdot \omega^5 + 0 \cdot \omega^6 \\
& \quad + \\
& 9 \cdot \omega^6 + 26 \cdot \omega^0 + 50 \cdot \omega^2 + 44 \cdot \omega^4 + 33 \cdot \omega^6 + 18 \cdot \omega^0 + 0 \cdot \omega^2 + 0 \cdot \omega^4 \\
& \quad + \\
& 9 \cdot \omega^5 + 26 \cdot \omega^0 + 50 \cdot \omega^3 + 44 \cdot \omega^6 + 33 \cdot \omega^1 + 18 \cdot \omega^4 + 0 \cdot \omega^7 + 0 \cdot \omega^2 \\
& \quad + \\
& 9 \cdot \omega^4 + 26 \cdot \omega^0 + 50 \cdot \omega^4 + 44 \cdot \omega^0 + 33 \cdot \omega^4 + 18 \cdot \omega^0 + 0 \cdot \omega^4 + 0 \cdot \omega^0 \\
& \quad + \\
& 9 \cdot \omega^3 + 26 \cdot \omega^0 + 50 \cdot \omega^5 + 44 \cdot \omega^2 + 33 \cdot \omega^7 + 18 \cdot \omega^4 + 0 \cdot \omega^1 + 0 \cdot \omega^6 \\
& \quad + \\
& 9 \cdot \omega^2 + 26 \cdot \omega^0 + 50 \cdot \omega^6 + 44 \cdot \omega^4 + 33 \cdot \omega^2 + 18 \cdot \omega^0 + 0 \cdot \omega^6 + 0 \cdot \omega^4 \\
& \quad + \\
& 9 \cdot \omega^1 + 26 \cdot \omega^0 + 50 \cdot \omega^7 + 44 \cdot \omega^6 + 33 \cdot \omega^5 + 18 \cdot \omega^4 + 0 \cdot \omega^3 + 0 \cdot \omega^2
\end{aligned}$$

Now, let's look carefully at this. In the 26 column, we get  $\omega^0$  all the way down, so that column sum is  $8 \cdot 26 = 228$ . In the 9, 50 and 33 columns and the first 0 column, we get the previous sum

$$\omega^0 + \omega^1 + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7,$$

so these columns sum to zero. In the 44 and the second 0 column we have

$$\begin{aligned} & \omega^0 + \omega^2 + \omega^4 + \omega^6 + \omega^0 + \omega^2 + \omega^4 + \omega^6 \\ &= 1 + \omega^2 - 1 - \omega^2 + 1 + \omega^2 - 1 - \omega^2 \\ &= 0 \end{aligned}$$

and in the 18 column we have

$$\begin{aligned} & \omega^0 + \omega^4 + \omega^0 + \omega^4 + \omega^0 + \omega^4 + \omega^0 + \omega^4 \\ &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \\ &= 0. \end{aligned}$$

So what we get as the final sum for the second line entry in the column vector for  $h(x)$  is just the 228.

If we now look at the third line, we'll get something entirely similar to what just happened for the second line. Everything will sum to zero except the 50 column, and we'll get  $8 \cdot 50 = 400$  there.

Because we're doing the Fourier Transform with power-of-2 roots of unity, we'll get

$$F^{-1} \cdot hvec = \begin{pmatrix} 8 \cdot 9 \\ 8 \cdot 26 \\ 8 \cdot 50 \\ 8 \cdot 44 \\ 8 \cdot 33 \\ 8 \cdot 18 \\ 8 \cdot 0 \\ 8 \cdot 0 \end{pmatrix}$$

and when we divide out the 8, we get the coefficients for  $h(x)$ .