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Ex. 45 (Schöning)

$$F_1 \equiv \forall x P(x,x)$$
$$F_2 \equiv \forall x \forall y (P(x,y) \rightarrow P(y,x))$$
$$F_3 \equiv \forall x \forall y \forall z ((P(x,y) \wedge P(y,z)) \rightarrow P(x,z))$$

All interpretations we present have the same universe,
$U = \{1,2,3\}$ (domain (of discourse); universe)

$\mathcal{A}_1$ is a ~~structure (interpretation)~~ model for F_2 and F_3, but not for F_1: $P^{\mathfrak{a}_1} = \{\}$ $= \mathfrak{a}_1(P)$ (This interpretation maps P to the empty relation.)

$$F_1^{\mathfrak{a}_1} = \text{false } (=0), \quad F_2^{\mathfrak{a}_1} = F_3^{\mathfrak{a}_1} = \text{true } (=1)$$

$\mathfrak{a}_2$ is a ~~structure~~ model for F_1 and F_3, but not for F_2:

$$P^{\mathfrak{a}_2} = \{(1,1), (2,2), (3,3), (1,2)\}$$

$\mathfrak{a}_3$ is a model for F_1 and F_2, but not for F_3:

$$\{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$$

Ex. 46 (Predicate logic with equality)

Exercise 46: In predicate logic with *identity* the symbol $=$ is also permitted in formulas (as a special binary predicate with a fixed interpretation) which is to be interpreted as identity (of values) between terms. How has the syntax (i.e. the definition of formulas) and the semantics (the definition of $\mathcal{A}(F)$) of predicate logic to be extended to obtain the predicate logic with identity?

Syntax: if t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.

Semantics: if F has the form $t_1 = t_2$, then

$$\mathcal{A}(F) = \begin{cases} 1 & \text{if } \mathcal{A}(t_1) = \mathcal{A}(t_2) \\ 0 & \text{otherwise} \end{cases}$$

Exercise 47: Which of the following structures are models for the formula

$$F = \exists x \exists y \exists z \, (P(x,y) \wedge P(z,y) \wedge P(x,z) \wedge \neg P(z,x)) \ ?$$

(a) $U_{\mathcal{A}} = \mathbb{N}$, $P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m < n\}$ yes $\left\{ \begin{array}{l} x<y \\ x<z, \ z \geq x \end{array} \right.$ ✓

(b) $U_{\mathcal{A}} = \mathbb{N}$, $P^{\mathcal{A}} = \{(m, m+1) \mid m \in \mathbb{N}\}$

(c) $U_{\mathcal{A}} = 2^{\mathbb{N}}$ (the power set of $\mathbb{N}$),
$P^{\mathcal{A}} = \{(A,B) \mid A, B \subseteq \mathbb{N}, A \subseteq B\}$

(a) asks, in effect, whether there is a solution to the following system of inequalities (in the integers ≥ 0):

$$\begin{cases} x < y \\ z < y \\ x < z \\ (z \geq x) \end{cases} \qquad \begin{array}{l} x = 1 \\ y = 3 \\ z = 2 \end{array} \text{ is a solution}$$

(b) similarly, is there a solution in the non-negative integers to the following set of equalities?

$$\begin{cases} x = y+1 \\ z = y+1 \\ x = z+1 \\ (z \neq x+1) \end{cases}$$

there is no solution

$p^{a} = \{(0,1), (1,2), (2,3), (3,4), \ldots\}$

(the successor relation)

(c) $U^{a} = 2^{\mathbb{N}} = \{\{0\}, \{0,1\}, \{0,1,2\}, \ldots \{0,2\}, \ldots \{1,2\}, \ldots\}$

p^{a} = the (non-proper) subset relation

Is there a solution to this system?

$$\begin{cases} X \subseteq Y \\ Z \subseteq Y \\ (X \subseteq Z) \\ X \subset Z \end{cases}$$

$X = \{0\}$
$Y = \{0, 1, 2\}$
$Z = \{0, 1\}$

Exercise 48: Let F be a formula, and let $x_1, \ldots, x_n$ be the variables that occur free in F. Show:

(a) F is valid if and only if $\forall x_1 \forall x_2 \cdots \forall x_n F$ is valid,

(b) F is satisfiable if and only if $\exists x_1 \exists x_2 \cdots \exists x_n F$ is satisfiable.

This makes clear what the highlighted

part means.

If for a formula F and a suitable structure $\mathcal{A}$ we have $\mathcal{A}(F) = 1$, then we denote this by $\mathcal{A} \models F$ (we say, F is *true* in $\mathcal{A}$, or $\mathcal{A}$ is a *model* for F).

otherwise $\not\models F$. If there is at least one model for the formula F

Some authors (e.g. Yasuhara) call the mapping of variables to elements of the universe <u>assignment</u>. This exercise (part(a)) emphasises that a formula is valid if it is true for <u>all</u> assignments to the free variables.

Exercise 49: Find a closed satisfiable formula F, such that for every model $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ of F, $|U_{\mathcal{A}}| \geq 3$.

(read "E" as "equals" or "the same")

$\forall x\, E(x,x) \wedge \forall x \forall y \forall z \left[\neg E(x,y) \wedge \neg E(x,z) \wedge \neg E(y,z)\right]$

Exercise 50: Let F be a satisfiable formula and let $\mathcal{A}$ be a model for F with $|U_{\mathcal{A}}| = n$. Show that for every $m \geq n$ there is a model $\mathcal{B}_m$ for F with $|U_{\mathcal{B}_m}| = m$. Furthermore, there is a model $\mathcal{B}_\infty$ for F with $|U_{\mathcal{B}_\infty}| = \infty$.

Hint: Pick some element u from $U_{\mathcal{A}}$, and add new elements to $U_{\mathcal{B}_m}$ having the same properties as u.

$\mathcal{A} = (U_a, I_a)$ is a model. Define $\mathcal{B} = (U_B, I_B) = \mathcal{B}_m$ in this way:

$U_B = U_a \cup \{b_1, \ldots b_{m-n}\}$ (so that $|U_B| = m$)

We <u>widen</u> I_α to I_β.

includes b_i, b_j

if $(\ldots a, \ldots, a, \ldots) \in P^\alpha$, then $(\ldots, b_i, \ldots, b_j, \ldots) \in P^\beta$.

Similarly for function names

$$f^\beta(\ldots, b_i, \ldots, b_j \ldots) = f^\alpha(\ldots, a, \ldots a, \ldots)$$

$\beta_{[x/u]}(F) = \alpha_{[x/u]}(F)$ for all variables x, all formulas F and $u \in \{b_1, \ldots, b_{m-n}\}$.