2011-03-22

Note Title 2011-03-22

Normal Forms

Claim. If x is not free in G, then

$$\mathcal{Q}_{[x/u]}(G) = \mathcal{Q}(G)$$

Proof. $\mathcal{Q}_{[x/u]}$ is equal to $\mathcal{Q}$ except that x is assigned u.

If x is not free, then, regardless of the assignment of x (either $\mathcal{Q}$ or $\mathcal{Q}_{[x/u]}$), x is assigned

according to rule for $\forall$ or the rule for $\exists$.
So, the change in mapping for x does not matter. $\square$

$(\forall x\, F \vee \forall x\, G) \not\equiv \forall x\,(F \vee G)$ $\quad H(x) = F$

Exercise 55 $(\exists x\, F \wedge \exists x\, G) \not\equiv \exists x\,(F \wedge G)$ $\quad J(x) = G$

The suitable structure (interpretation) providing the required counterexample:

$U_{\mathcal{A}} = \{a, b\}$ $\quad \mathcal{A}(H)\,(= H^{\mathcal{A}}) = \{(a)\}$ $\quad \mathcal{A}(J)\,(= J^{\mathcal{A}}) = \{(b)\}$

$\mathfrak{a}(\forall x H(x) \vee \forall x J(x)) = 0$, but

$\mathfrak{a}(\forall x (H(x) \vee J(x))) = 1$, and

$\mathfrak{a}(\exists x H(x) \wedge \exists x J(x)) = 1$, but

$\mathfrak{a}(\exists x (H(x) \wedge J(x))) = 0$

<u>Exercise 56</u> Show that $F = (\exists x P(x) \to P(y))$ is equivalent to $G = \forall x (P(x) \to P(y))$.

[$P(x) \leftarrow Q(x,y)$ in Al Log]

$\forall x \forall y\, P(x) \leftarrow Q(x,y)$

$\equiv$

$\forall x\, (P(x) \leftarrow \exists y\, Q(x,y))$]

Let $\mathfrak{A} = (U_{\mathfrak{a}}, I_{\mathfrak{a}})$ a suitable structure (interpretation) for F and G.

$\mathfrak{A}(\exists x\, P(x) \to P(y)) = 1$

iff $\mathfrak{A}(\sim \exists x\, P(x) \vee P(y)) = 1$ (defn. of implication)

iff either $\mathfrak{A}(\sim \exists x\, P(x)) = 1$ or $\mathfrak{A}(P(y)) = 1$ (4. on p. 47

iff $\mathfrak{A}(\forall x \sim P(x))$ or $\mathfrak{A}(P(y)) = 1$ (1b p. 52)

iff $\mathfrak{A}(\forall x \sim P(x) \vee P(y)) = 1$ (4 on p. 47)

iff $\mathfrak{A}(\forall x (\sim P(x) \vee P(y))) = 1$ (2b p. 52 with $F = \sim P(x)$ and $G = P(y)$)

iff $\mathfrak{A}(\forall x (P(x) \Rightarrow P(y))) = 1$ (defn of implication) ;1

It would be shorter to provide $\mathfrak{A}$ directly?