2011-02-08

Note Title 2011-02-08

The Compactness Theorem for the propositional calculus (Section 1.4 Schöning, handout).

The proof below is by Loveland [Theorem 1.2.3] in:

Loveland, Donald W.

Automated Theorem Proving: A Logical Basis.

North-Holland, 1978.

Thm. A set S of formulas is satisfiable iff every finite subset T of S is satisfiable

Proof.

It is immediate that, if S is a satisfiable set of formulas, then any subset of S is a satisfiable set.

To show the converse, we show that if S is unsatisfiable, then there is a finite subset T of S that is also unsatisfiable.

Assume that S is unsatisfiable.

$\vdash (S \supset (\sim (A \supset A)))$ b/c $(S \supset (\sim (A \supset A)))$ is a tautology and the prop. calculus is complete (Thm 9.5 Vesc hara)

$S \vdash (\sim (A \supset A))$ converse of the deduction theorem

$T \vdash (\sim (A \supset A))$ for some finite $T \subset S$ b/c proofs/derivations are finite and so, we can choose as hypotheses only the formulas in S that are actually used in the proof.

$\vdash (T \supset (\sim (A \supset A)))$ deduction theorem

T is unsatisfiable, since $(\sim (A \supset A))$ is a contradiction and by the soundness of the prop. calculus $(T \supset (\sim A \supset A))$ is a tautology

$\square$

Examples of resolvents

$C_1 = \{A, \sim B\}$

$C_2 = \{B\}$

$$\begin{tikzcd}[column sep=small]
C_1 \arrow[dr, dash] & & C_2 \arrow[dl, dash] \\
& R &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\{A, \sim B\} \arrow[dr, dash] & & \{B\} \arrow[dl, dash] \\
& A &
\end{tikzcd}$$

$C_1 = \{A, \sim B\}$

$C_2 = \{B, \sim C\}$

$C_3 = \{C\}$

$$\begin{tikzcd}[column sep=small]
\{A, \sim B\} \arrow[dr, dash] & & \{B, \sim C\} \arrow[dl, dash] & \\
& \{A, \sim C\} \arrow[dr, dash] & & \{C\} \arrow[dl, dash] \\
& & A &
\end{tikzcd}$$

C_1, C_2, C_3 as just above,

$C_4 = \{\sim A\}$

$$\begin{tikzcd}[column sep=small]
\{A, \sim B\} \arrow[dr] & & \{B, \sim C\} \arrow[dl] & & \\
& \{A, \sim C\} \arrow[dr] & & \{C\} \arrow[dl] & \\
& & \{A\} \arrow[dr] & & \{\sim A\} \arrow[dl] \\
& & & \Box &
\end{tikzcd}$$

Exercise 30

$F = \{\{A, \sim B, C\}, \{B, C\}, \{\sim A, C\}, \{B, \sim C\}, \{\sim C\}\}$

$\vdash$ Res.

$\{A, \sim B, C\}$ $\{B, C\}$

$$Res_1 = \{ \{A, C\}, \{\sim B, C\}, \{A, C, \sim C\}, \{A, B, \sim B\}, \{A, \sim B\}, \{B\} \cup Res_0 \}$$

$$Res_2 = \{ \{A, C\}, \{C\}, \{A, B\}, \{A\}, \ldots \}$$

Recall:

$K \models G$ iff $\sim G \wedge K$ is unsatisfiable

$\sim G \cup K$ is unsatisfiable

$K \models G$ iff $K \vdash G$ iff $\vdash K \supset G$

soundness & completeness of the p.c.

deduction theorem and its converse

$K \models G$ iff $K \vdash G$ iff $\vdash K \supset G$ iff

iff $\models K \supset G$ iff $\models \neg K \vee G$ iff

equivalent formulas

$\sim(\sim K \vee G)$ is a contradiction (i.e. it is unsatisfiable)

iff $K \wedge \sim G$ is unsatisfiable

$\vdash G$: there is a proof (or derivation) of G

$\models G$: G is a tautology

$K \vdash G$: there is a proof of G from hypothesis K

$K \models G$: K is a model of G

in every interpretation in which K holds, G also holds

$\vdash K \supset T$ There is a proof of $K \supset T$

If $K \vdash G$ then $K \models G$ is soundness (Thm 9.2 for the P. C.)

If $K \models G$ then $K \vdash G$ is completeness (Thm. 9.5)

Memo trick: $\vdash \;\Rightarrow\; \models$

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