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Horn formulas

Consider a disjunction of literals (a clause in a formula in CNF)

Recall: A generic CNF formula F is a conjunction $\wedge$ of clauses, where each clause is a disjunction of literals:

$$F = \bigwedge_{i=1}^{m} \left(\bigvee_{j=1}^{k_m} L_{i,j} \right)$$

A disjunction of n literals can be written in implicative form as follows:

$A_1 \wedge \dots \wedge A_k \to B_1 \vee \dots \vee B_\ell$, which is equivalent to

$\neg(A_1 \wedge \dots \wedge A_k) \vee (B_1 \vee \dots \vee B_\ell)$, which is eq. to

$\neg A_1 \vee \neg A_2 \vee \dots \vee \neg A_k \vee B_1 \vee \dots \vee B_\ell$

A Horn clause is a (disjunctive) clause in implicative form with at most one positive literal.

Horn clauses may be divided into three types:

(1) A_i (facts, sometimes written $(1 \to A_i)$)

(2) $A_1 \wedge \dots \wedge A_k \to B$ (rule)

(3) $\sim A_1 \vee \dots \vee \sim A_k$ (integrity constraint, sometimes written $(A_1 \wedge \dots \wedge A_k \to 0)$)

Schöning defines a Horn formula F to be a conjunction of Horn clauses

Satisfiability algorithm for a Horn formula F.
(F is a conjunction of Horn clauses; F is a conjunction of the formulas in a Horn knowledge Base ("Horn KB").)

1. Mark every occurrence of an atomic formula A in F if there is a subformula of the form $(1 \to A)$ in F.
("Mark the facts in the Horn KB.")

2. **while** there is a subformula G in F of the form $(A_1 \wedge \dots \wedge A_n \to B)$ or of the form $(A_1 \wedge \dots \wedge A_n \to 0)$, $n \geq 1$, where $A_1, \dots, A_n$ are already marked (and B is not yet marked)

do if G is the first form then mark B
else output 'unsatisf.' and halt.

3. Output 'satisfiable' and halt.
(The satisfying assignment (model) is given by the marking in this way: $\mathcal{A}(A_i) = 1$ iff A_i is marked.)

Theorem: The above marking is correct for Horn formulas and stops after at most n applications of the while loop, where n is the number of subformulas in F (i.e., the number of clauses in the Horn KB).

Proof

(Termination in at most n iterations)

During each iteration either the algorithm stops or it marks the positive literal in one subformula (clause; more specifically, rule). There are only n clauses.

(Correctness)

(1) Claim: Every model $\mathcal{A}_i$ of F must satisfy $\mathcal{A}(A_i) = 1$ for each atomic formula A_i of F marked by the algorithm.

Obvious for subformulas (clauses) $(1 \to A)$ (i.e., the facts A), b/c F is a conjunction of clauses, and a conjunction is true iff every one of its conjuncts is true

In step 2, B in $(A_1 \wedge \cdots \wedge A_n \to B)$ is marked if $A_1, \ldots, A_n$ are marked. Since $A_1, \ldots A_n$ are true, then $A_1 \wedge \cdots \wedge A_n \to B$ is true only if B is also true.

(2) The decision for "unsatisfiable" in step 2 is correct if there is a clause $A_1 \wedge \cdots \wedge A_n \to 0$ with $A_1, \ldots, A_n$ already marked b/c it cannot be that $A_1, \ldots, A_n$ are all true but $A_1 \wedge \cdots \wedge A_n \to 0$ is also true.

(3) If the marking process ends successfully, i.e. step 3 is reached, then F is satisfiable and the marking provides a model for F.

To show this, consider the generic clause of F (call it G)

If G has the form $(1 \to A)$ $(\equiv A)$, then step 1 takes care of it.

If G has the form $(A_1 \wedge \cdots \wedge A_n \to B)$ then either

(a) all of $A_1 \ldots A_n$ are marked and then B is also marked in step 2. Therefore, G is true in the model based on the assignment b/c $t \to t$ is t, or

(b) one of $A_1, \ldots A_n$ is not marked. Then $A_1 \wedge \cdots \wedge A_n \to B$ is true b/c $f \to t$ is t or $f \to f$ is t.

If G has the form $(A_1 \wedge \ldots \wedge A_n \to 0)$ then, by the assumption that step 3 is reached, for at least one of $A_1, \ldots, A_n$, (say A_i),

A_i is not marked, and G is true in the interpretation provided by the marking, b/c $f \to f$ is t.

So, the marking provides a model for <u>each</u> of the subformulas of F (clauses of the Horn KB) and therefore a model of F (of the whole Horn KB).

Exercise 21.

$F = (\sim A \vee \sim B \vee \sim D) \wedge \sim E \wedge (\sim C \vee A) \wedge C \wedge B \wedge (\sim G \vee D) \wedge G$

$$KB = \left\{ \begin{array}{l} \sim A \vee \sim B \vee \sim D \\ \sim E \\ C \to A \\ C \\ B \\ G \to D \\ G \end{array} \right\} \quad || \quad \left\{ \begin{array}{l} A \wedge B \wedge D \to 0 \\ E \to 0 \\ \\ 1 \to C \\ 1 \to B \\ \\ 1 \to G \end{array} \right\} \quad \begin{array}{l} \text{unsatisfiable} \\ \\ A \\ \checkmark \\ \checkmark \\ D \\ \checkmark \end{array}$$

A Horn KB that does not contain integrity constraints is a definite clause KB.

Observations: (1) every definite clause KB has a model (assign 1 (t) to every atomic formula)

(2) every Horn KB without facts has a model (assign 0 (f) to every atomic formula).