CSCE 551
Midterm Exam II
Thursday April 7, 2005
Answers

This test is open book, open notes, but no electronic devices. Do all problems, putting your answers in the sheets provided. There are 100 points total in the exam. For graduate students, 90 points constitute full credit, with 10 points extra credit. For undergrads, 70 points constitute full credit and the other 30 are extra credit. You have 75 minutes. Please read a question *carefully* before attempting it. If you have any questions or doubts about what is expected, please ask me.

1. (25 points) Let

   $$L = \{\langle R, S\rangle \mid R \text{ and } S \text{ are regular expressions and } L(R) \cap L(S) = \emptyset\}.$$

   Show that L is decidable by giving a high-level decision procedure for L.

   Answer: Let M be the following TM:
   "On input $\langle R, S\rangle$ where R and S are regular expressions:

   (a) Construct DFAs A and B equivalent to R and S, respectively.
   (b) Using the Cartesian product construction, construct a DFA C such that $L(C) = L(A) \cap L(B)$.
   (c) Run the decider for the language E_{DFA} on input $\langle C\rangle$."

   M is clearly a decider. Further, we have, for all regular expressions R and S,

   $$\begin{aligned} \langle R, S\rangle \in L &\iff L(R) \cap L(S) = \emptyset \\ &\iff L(A) \cap L(B) = \emptyset \\ &\iff L(C) = \emptyset \\ &\iff \langle C\rangle \in E_{\text{DFA}} \\ &\iff M \text{ accepts } \langle R, S\rangle. \end{aligned}$$

   Thus M decides L.

2. (25 points) Recall that ε is the empty string. Let

$$L = \{\langle M\rangle \mid M \text{ is a TM and } M \text{ rejects } \varepsilon\}.$$

Show that L is undecidable by giving a mapping reduction from A_{TM} to L.

Answer: Let f be the function computed by the following transducer:
"On input $\langle M, w\rangle$ where M is a TM and w is a string:

(a) Let $R =$

'On input x:

i. Run M on input w.

ii. If M ever accepts w, then reject; if M ever rejects w then accept.'

(b) Output $\langle R\rangle$."

The function f defined above is clearly computable. Given TM M and string w, we have $f(\langle M, w\rangle) = \langle R\rangle$, where R is constructed in step (a). Clearly, if M accepts w, then R rejects all strings, including ε. Otherwise, if M does not accept w, then R rejects no strings. Thus we have $\langle M, w\rangle \in A_{\text{TM}}$ if and only if $f(\langle M, w\rangle) \in L$, and so f m-reduces A_{TM} to L.

3. (20 points) Let

$$L = \{\langle A\rangle \mid A \text{ is an NFA and } L(A) \text{ is infinite}\}.$$

Show that L is decidable in polynomial time (that is, in the class $\mathbf{P}$) by giving a polynomial-time decision procedure for L.

Answer: Let M be the following TM:
"On input $\langle A\rangle$ where A is an NFA:

(a) If A is not clean, then make it so.

(b) By doing a breadth-first search from A's start state, remove all states from A that are unreachable from the start state. If the accepting state is removed, then reject.

(c) By doing a backwards breadth-first search from A's accepting state, remove from A any states from which the accepting state cannot be reached.

(d) By doing a topological sort on the remaining transition graph of A, look for a cycle. (Any other kind of search will also work.)

(e) If a cycle was found, then accept; otherwise reject."

The algorithm above is clearly polynomial time, and M decides L because $L(A)$ is infinite iff its transition diagram contains a cycle somewhere along some path from the start state to the accepting state.

4. (30 points total)

(a) (20 points) A function $f : \Sigma^* \to \Sigma^*$ is *honest* if there is a natural number k such that for all $x \in \Sigma^*$,

$$|x| \leq |f(x)|^k.$$

That is, f is honest if it does not map really long strings to really short strings. Suppose that f is polynomial-time computable and honest. Show that the range of f is in **NP**, where the range of f is defined as $\{f(x) \mid x \in \Sigma^*\}$.

Answer: Since f is honest, there is a k such that $|x| \leq |f(x)|^k$ for all strings x. Fix such a k. Here is a polynomial-time verifier V for the language $\{f(x) \mid x \in \Sigma^*\}$, that basically checks whether its proof c is such that $f(c) = w$:
"On input $\langle w, c\rangle$ where w and c are strings:

i. If $|c| > |w|^k$, then reject. (c is too long to be a preimage of w in this case.)

ii. Compute $y = f(c)$. (Since f is polynomial time, this can be done in time polynomial in $|c|$. Since $|c| \leq |w|^k$, this is also polynomial time in $|w|$.)

iii. If $y = w$, then accept; else reject.

V on input $\langle w, c\rangle$ evidently runs in time polynomial in $|w|$, so V is a polynomial-time verifier. We have that a string w is in the range of f iff there is some c such that $f(c) = w$, iff (since f is honest) there is some c with $|c| \leq |w|^k$ such that $f(c) = w$, iff there is some c such that V accepts $\langle w, c\rangle$. Therefore, the range of f is in **NP** via the verifier V.

(b) (10 points) Describe a polynomial-time computable, honest function whose range is equal to SAT.

Answer: Let g be the function computed by the following transducer T:

"On input w:

i. Let $n = |w|$, and define

$$\varphi_n := x_1 \wedge x_2 \wedge \cdots \wedge x_n.$$

(Clearly, φ_n is satisfied by setting all its variables to TRUE. The formula φ_n is our "fall-back" formula.)

ii. If w is not of the form $\langle \varphi, a \rangle$ where φ is a Boolean formula and a is a binary string encoding some truth assignment to the variables of φ, then output $\langle \varphi_n \rangle$.

iii. Let $w = \langle \varphi, a \rangle$ as above. If a does not satisfy φ (that is, φ is FALSE under the truth assignment a), then output $\langle \varphi_n \rangle$.

iv. Otherwise, output $\langle \varphi \rangle$.

Clearly, g is polynomial-time computable and honest, since its output is always at least (roughly) half the length of its input. Further, g only outputs satisfiable formulas, so its range is a subset of SAT. Finally, for any $\langle \varphi \rangle \in$ SAT, there is some a that satisfies φ, and so for this a, we have

$$\langle \varphi \rangle = g(\langle \varphi, a \rangle).$$

So the range of g is exactly SAT.