CSCE 551
Final Exam
Saturday May 1, 2004

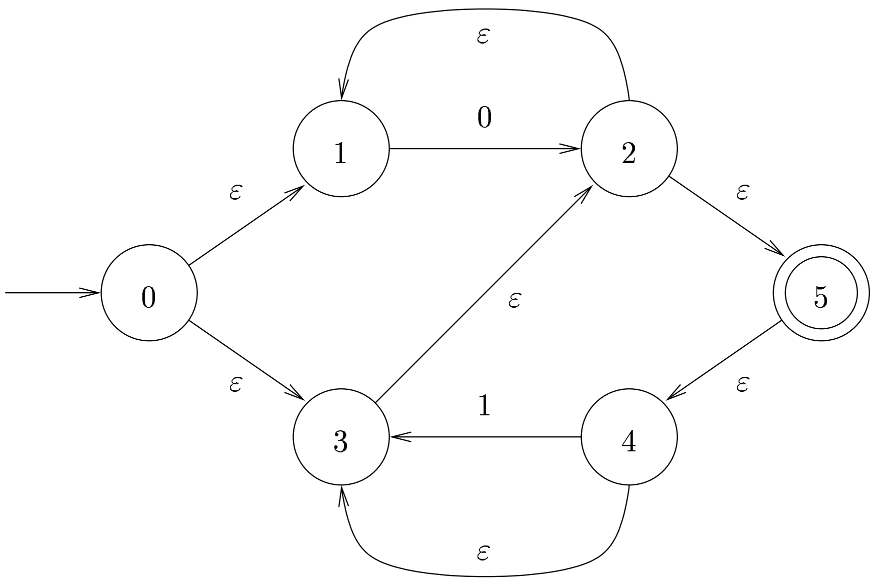
Do all problems, putting your answers on the blank sheets provided. There are 90 points total in the exam. For graduate students, a total of 80 points is considered full credit, and the rest is extra credit. For undergrads, 60 points is considered full credit and the rest is extra credit. All questions are worth 10 points but they may not have the same difficulty. If you have any questions, please ask the proctor. You have 180 minutes.

Please note the following:

- This exam is open book, open notes, but no electronic devices.
- In general, unless I say otherwise, you need not show your scratch work to get full credit for a correct answer. If your answer is incorrect, however, showing your work may earn you partial credit.
- We can encode natural numbers $N = \{0, 1, 2, \ldots\}$ by strings over the alphabet $\Sigma = \{0, 1\}$ via the standard binary representation.
- Recall that
$$A_{\mathrm{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts input } w\}.$$
You may assume as given that A_{TM} is Turing-recognizable but undecidable.
- You may also take as given any fact proved in the textbook.
- The complement of a language L is denoted by $\overline{L}$.
- A Turing machine either accepts, rejects, or loops on its input. The first two are halting behaviors.

One more thing! Please fill out the course evaluation sheet attached to your exam. It will only take a couple of minutes. Put your name and Student ID Number on it and return it to the proctor when you hand your exam in. It is important that you put your name and Student ID Number on the sheet. The proctor will hand it in directly to the department office without me seeing it (and I am solely responsible for the grades). This will be used for long-term curriculum development and improvement and will not affect your grade in any way.

1. (10 points) Using any method you like (including intuition), give the unique minimal DFA equivalent to the following NFA:



If your answer is correct, you get full credit even if you do not show how you arrived at it.

2. (10 points) Give an implementation-level description of a standard (1-tape deterministic) TM M that decides the following language over input alphabet $\{0, 1\}$:

$$\{w \mid w \text{ contains at least as many zeros as ones}\}.$$

3. (10 points) Let A and B be two disjoint languages. Recall (Problem 4.18) that a language C *separates* A from B if $A \subseteq C$ and $B \subseteq \overline{C}$. Consider the two languages

$$A_{\text{DIAG}} = \{\langle M \rangle \mid M \text{ is a TM that accepts the string } \langle M \rangle\}$$

and

$$R_{\text{DIAG}} = \{\langle M \rangle \mid M \text{ is a TM that rejects the string } \langle M \rangle\}.$$

By filling in the bracketed parts, complete the following proof that there is no decidable language C that separates A_{DIAG} from R_{DIAG}:

> Suppose that there *is* some decidable C separating A_{DIAG} from R_{DIAG}. Let D be the following machine: "On input w: [YOU FILL IN THIS PART (HIGH-LEVEL DESCRIPTION)]." Consider D running on input $\langle D \rangle$. Clearly, D does not loop on input $\langle D \rangle$. But, if D accepts $\langle D \rangle$, then [YOU EXPLAIN WHY D MUST REJECT $\langle D \rangle$]. Likewise, if D rejects $\langle D \rangle$, then [YOU EXPLAIN WHY D MUST ACCEPT $\langle D \rangle$]. This is a contradiction, thus there is no such decidable C.

4. (10 points) Let f be any computable function. Show that the set

$$R = \{y \mid (\exists x \in A_{\mathrm{TM}})\ f(x) = y\}$$

is Turing-recognizable by giving a high-level description of a TM that recognizes R.

5. (10 points) Find a mapping reduction from A_{TM} to the language

$$\{\langle M\rangle \mid M \text{ is a TM and } L(M) = A_{\mathrm{TM}}\}.$$

6. (10 points) Show that there is no computable function f outputting natural numbers such that, for any TM M and string w, if M accepts w, then M accepts w in at most $f(\langle M, w\rangle)$ steps. [Hint: Argue by contradiction.]

7. Below, G is always an undirected graph, and k is a natural number. A path in G is *simple* if no vertex appears more than once along the path. The *length* of a path is the number of edges in the path.

   (a) (10 points) Explain why the language

   $$\text{LONGPATH} = \{\langle G, v, k\rangle \mid G \text{ has a simple path of length } k \text{ starting at vertex } v\}$$

   is in NP.

   (b) (10 points) Show that if LONGPATH $\in$ P, then there is a polynomial-time computable function f that, on input $\langle G, k\rangle$, either outputs some simple path in G of length k or outputs "no" if there is no such path. You may take the statement of part (a) as given. A high-level description of f is fine.

8. (10 points) Using any method you like, show that the language

$$\{0^m 1^n \mid m, n \geq 0 \text{ and } n \neq m^2\}$$

is not regular.