CSCE 355
3/27/2024

Project (briefly)
Pumping Lemma for CFLs.

Project: (1) ε-NFA ⟶ NFA

method given in class

3 steps:

1. Back-propagate accepting states;

2. Back-propagate non-ε-transitions;

3. Remove all ε-transitions

(2) Simulate an NFA on various input strings (sets-of-states method)

standard input

Pumping Lemma for CFL's

~~thm~~ Used to show that a language is not context-free.

Definition: A language L is *CFL-pumpable* if there exists a $p > 0$ (the pumping length),

for all $s \in L$ with $|s| \geq p$,

there exist strings v, w, x, y, z such that

1) $s = vwxyz$,
2) $|wxy| \leq p$,
3) $|wy| > 0$ (w & y are not both ε)

and

for every $i \geq 0$, $vw^ixy^iz \in L$.

Lemma (Pumping Lemma for CFLs):

Every context-free language is CFL pumpable.
[Proof later]

Technique: show a language is not a CFL via showing it is not CFL-pumpable.

L is not CFL-pumpable means

$$(\forall p > 0)(\exists s \in L, |s| \geq p)\left(\forall v,w,x,y,z, \begin{array}{l} s = vwxyz \\ |wxy| \leq p \\ |wy| > 0 \end{array}\right)$$

$$(\exists i \geq 0)\left[vw^ixy^iz \notin L\right]$$

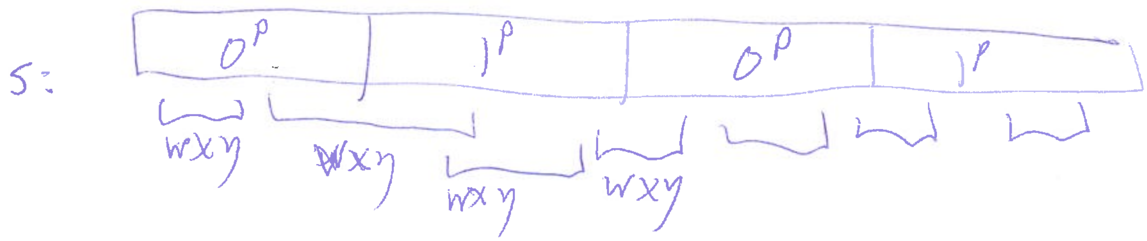
Ex: Let $L := \{ww : w \in \{0,1\}^*\}$

Claim: L is not CFL-pumpable.

Proof: Given $p > 0$, let $s := 0^p 1^p 0^p 1^p$

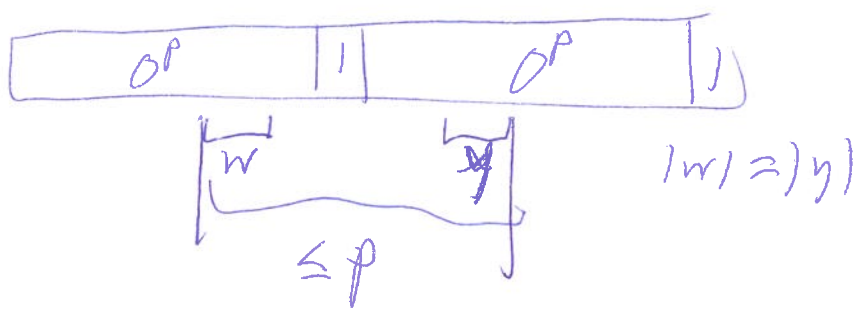
Given $v, \underbrace{w, x, y}, z$ as above, set $i := 0$ $(vw^0xy^0z = vxz$

Claim that $vxz \notin L$:



Case by case analysis. //

An s that doesn't work: $s = 0^p 1 0^p 1$



Ex: $L = \{a^j b^j c^j : j \geq 0\}$

L not CFL pumpable:

Given $p > 0$, let $s := a^p b^p c^p$.

Given v, w, x, y, z such that (1–3) hold,

let $i := 0$. Then $vxz \notin L$:

wxy can't contain both a's and c's,
so wxz has (fewer a's or fewer b's but same # of c's) or (fewer b's or c's but same # a's). either way $vxz \notin L$.

Similar example:

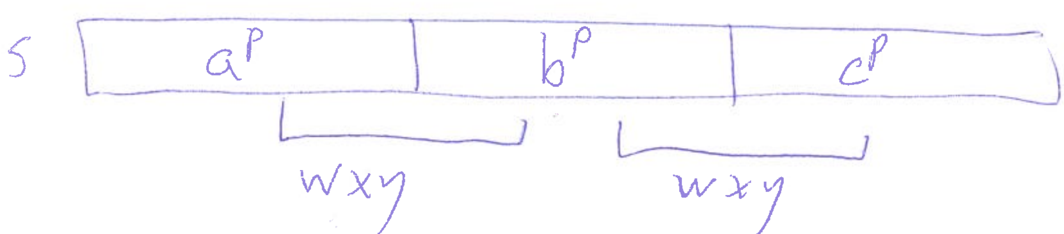
$$L := \{ a^j b^k c^\ell : 0 \le j \le k \le \ell \}$$

L is not CFL-pumpable:

Given p: let $s := a^p b^p c^p \in L$.

Given v, w, x, y, z satisfying (1–3),

$$\text{Let } i := \begin{cases} 2 & \text{if } wxy \text{ contains no } c\text{'s, else} \\ 0 & \text{if } wxy \text{ contains no } a\text{'s} \end{cases}$$



Note:
wxy cannot contain both a's and c's because $|wxy| \le p$ & a's & c's are separated by p many b's.

Then $vw^ixy^iz \notin L$ in either case. //

Try yourself: $L := \{a^m b^n c^m d^n : m, n \geq 0\}$

is not CFL pumpable.

<u>Proof</u> of the Pumping Lemma for CFL's:

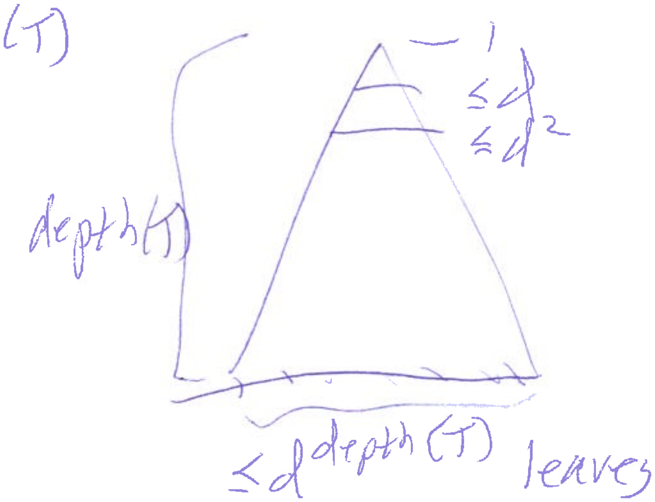
Fix a CFL L. Fix a CFG G for L. Let n be the number of nonterminals of G, and let

$$d := \max\{|\alpha| : \alpha \text{ is the body of some production of } G\}$$

Let $p := d^{n+1}$.

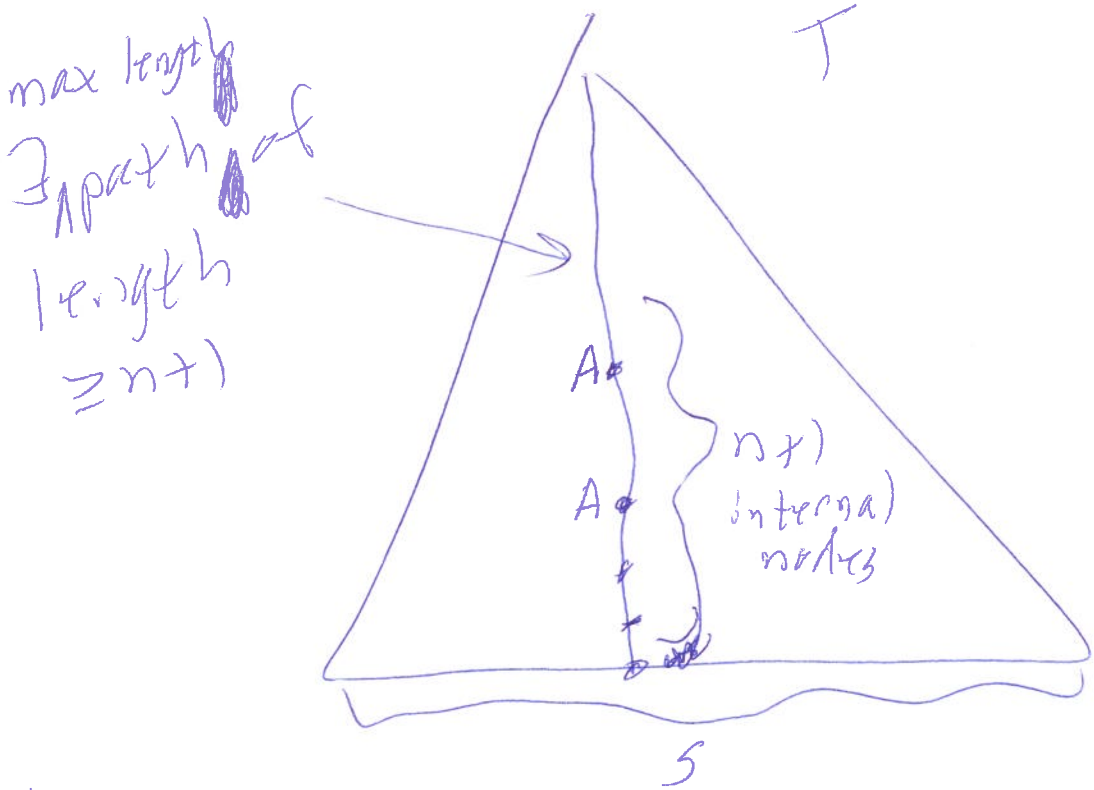
Let s be any string in L of length $\geq p$.

Let T be a parse tree of minimum size yielding s. Every internal node of T has $\leq d$ many children, so # leaves of T is at most $d^{\text{depth}(T)}$

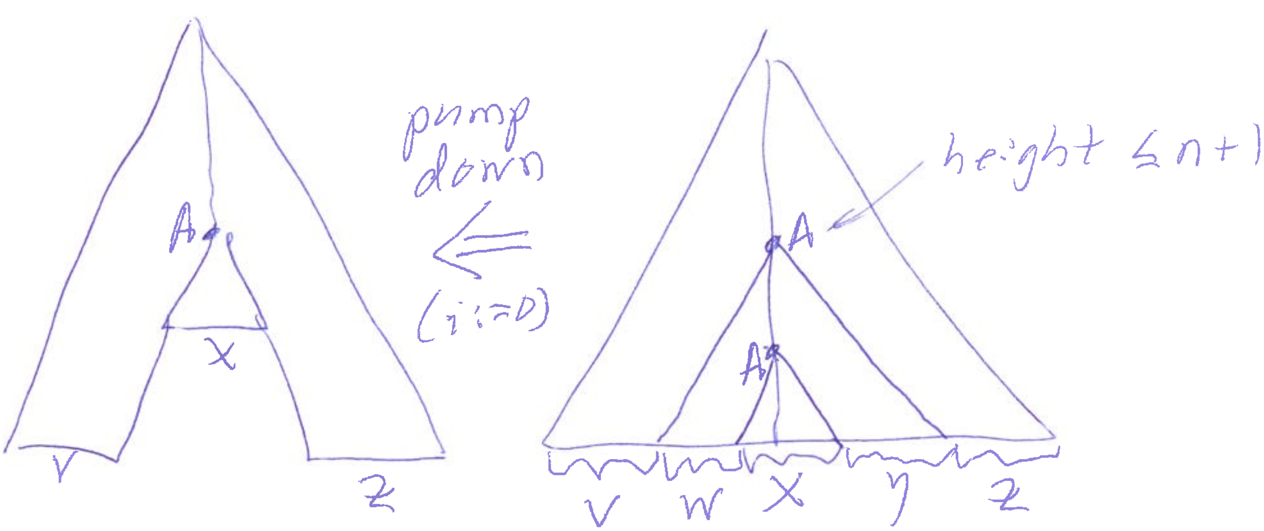


$|s| \geq p$, so T has $\geq p = d^{n+1}$ many leaves,

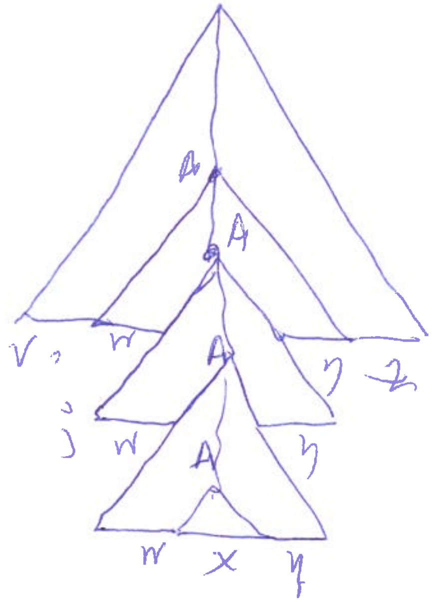
$\therefore$ depth$(T) \geq n+1$



only n many vars in G, so some var, say A, appears twice among the bottom $n+1$ nodes of path:



$\Downarrow$ pump up:

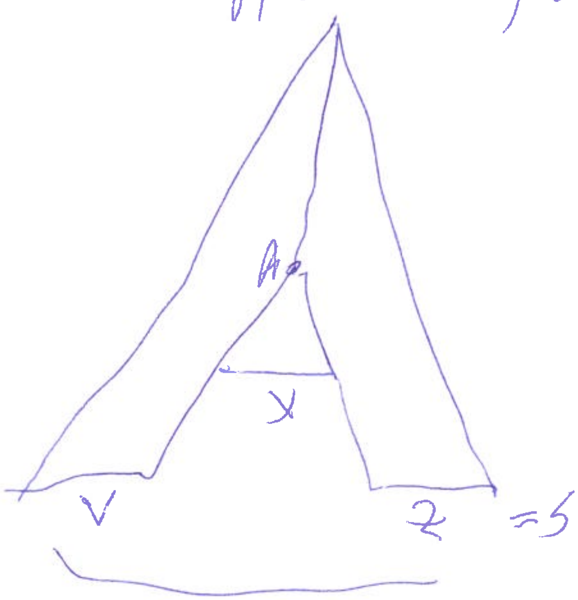


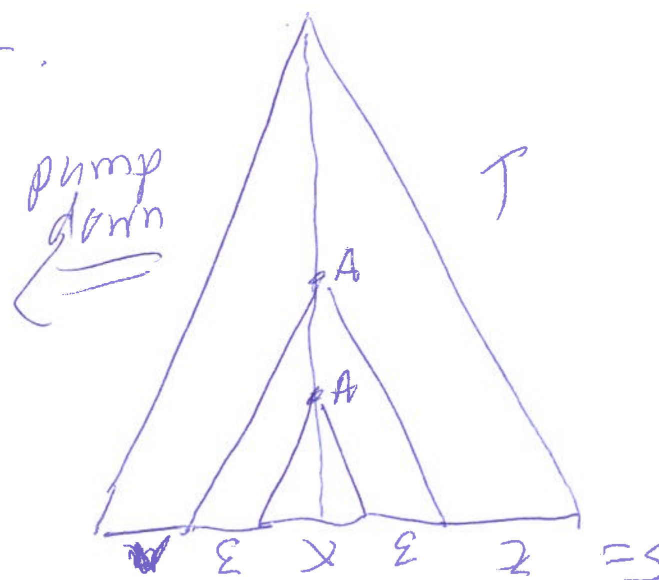
Every tree is a parse tree for a string in L.

Must check $|wxy| \le p$ b/c each A has height $\le n+1$

Why $|wy| > 0$?

Suppose $w = y = \varepsilon$.





smaller tree yielding s ↯

(T was minimum)