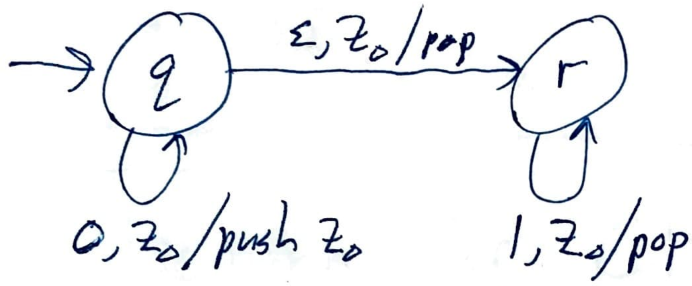
① Restricted PDA P such that $N(P) = \{0^n 1^n : n \geq 0\}$

CSCE 355
4/6/2022



Construct equivalent grammar: $\langle V, \{0,1\}, S, Prod \rangle$

$$V = \{S, \underset{A}{[qZ_0q]}, \underset{B}{[qZ_0r]}, \underset{C}{[rZ_0q]}, \underset{D}{[rZ_0r]}\}$$

$S \to A \mid B$

$A \to 0AA \mid 0BC \mid \varepsilon$

$B \to 0AB \mid 0BD$

$C \to \cancel{1CA} \mid \cancel{1DC} \quad 1$

$D \to \cancel{1CB} \mid \cancel{1DD} \quad 1$

pop transitions

$S \to A \mid B$

$B \to \varepsilon$

$D \to 1$

push trans

$A \to 0AA \mid \underbrace{0BC}_{\text{omit}}$

$B \to 0AB \mid 0BD$

~~B productions are useless, can be omitted~~

Simpler grammar:

$S \to A \mid B$

$B \to \varepsilon$

$D \to 1$

$A \to 0AA$

$B \to 0AB \mid 0BD$

simplify $\Longrightarrow$

$S \to \overbrace{A}^{\text{omit}} \mid B$

$B \to \varepsilon$

$A \to 0AA$ } omit

$B \to \underbrace{0AB}_{\text{omit}} \mid 0B1$

② simplify $\Longrightarrow$

$$S \to B$$
$$B \to \varepsilon$$
$$B \to 0B1$$

$\xRightarrow{\text{simplify}}$ $B \to 0B1 \mid \varepsilon$

Pumping Lemma for CFLs

Use to show that $\{0^n 1^n 2^n : n \geq 0\}$ is not a CFL

Lemma (Pumping Lemma for ~~CFL~~ CFLs):
Let L be a CFL. Then there exists a $p > 0$ (the pumping length) such that, for ~~every~~ every $s \in L$ such that $|s| \geq p$, there exist strings u, v, w, x, y such that

a) $s = uvwxy$,

$\to$ b) $|vwx| \leq p$, and

c) $|vx| > 0$ (v and x cannot both be ε)

and for all $i \geq 0$, $uv^i wx^i y \in L$.

Say that L is CFL-pumpable if it satisfies the conclusion of the lemma. [$\therefore$ every CFL is CFL-pumpable]

(3) Application: $L = \{0^n 1^n 2^n : n \geq 0\}$ is not ~~CFL~~ CFL-pumpable ($\therefore$ not context-free).

L is not CFL-pumpable means:

$\forall p > 0,\ \exists s \in L,\ |s| \geq p,$

$\forall u, v, w, x, y$ such that (a), (b), (c) above,

$\exists i \geq 0$ such that $uv^iwx^iy \notin L$.

Proof that the L above is not CFL-pumpable:

Given $p > 0$, let $s := 0^p 1^p 2^p$ $\quad (s \in L,\ |s| = 3p \geq p)$

Given u, v, w, x, y such $s = uvwxy$, $|vwx| \leq p$, $|vx| > 0$,

let $i := 0$. This works:

Since $|vwx| \leq p$, either vwx has ~~not~~ no 2's or has no 0's. So, letting $t := uv^0wx^0y = uwy$, t either has p many 2's & fewer of either 1's or 0's (or both), or has p many 0's & fewer 1's or 2's (or both).

In any case, $t = uwy \notin L$.

$\therefore$ L is not CFL-pumpable

$\therefore$ L is not a CFL (by P.L. for CFLs).

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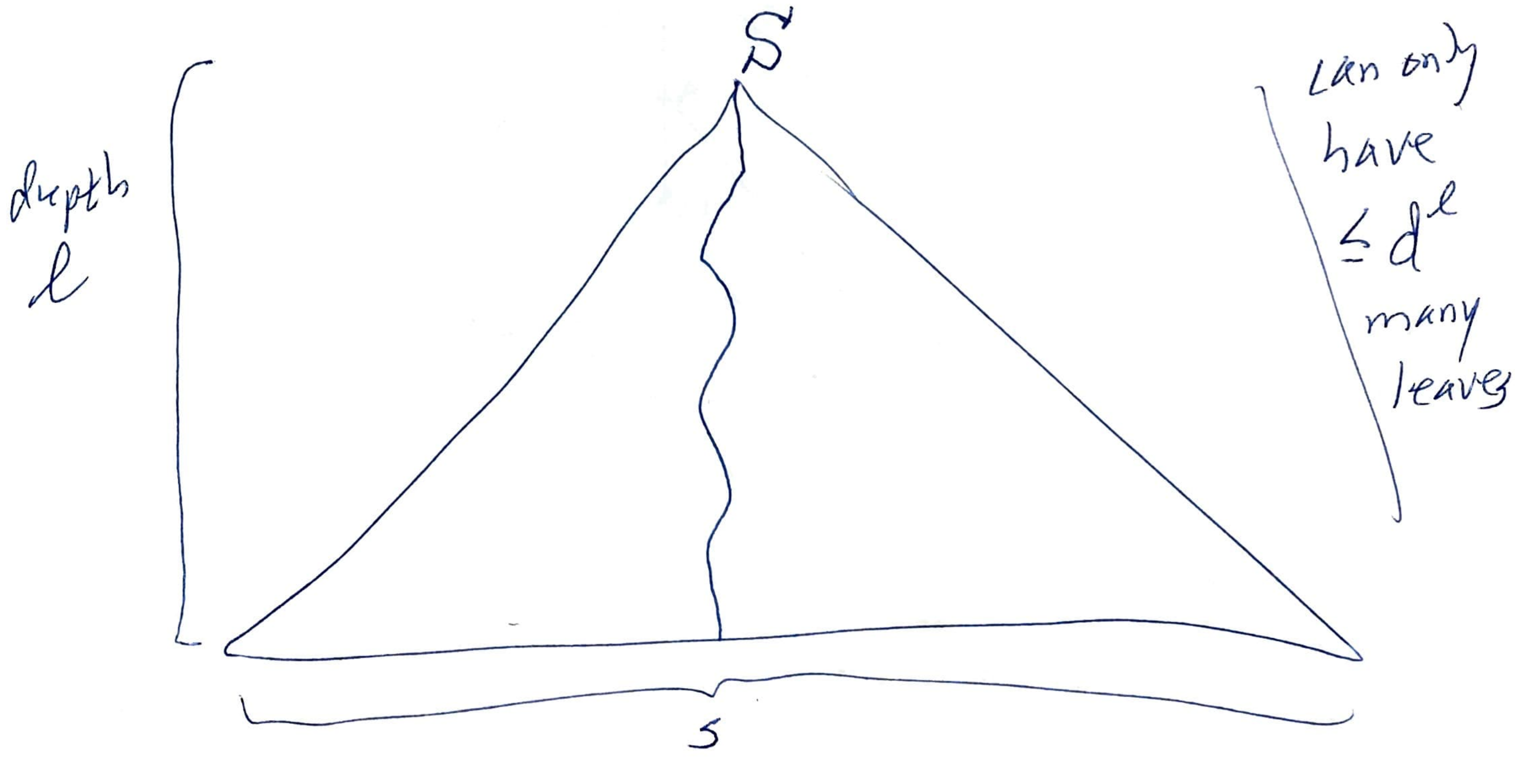
④ Proof of the Pumping Lemma for CFL's:

Let L be a context-free language. Can let $G := \langle V, \Sigma, S, P \rangle$ be a CFG such that $L = L(G)$. Such a G exists, because L is a CFL.

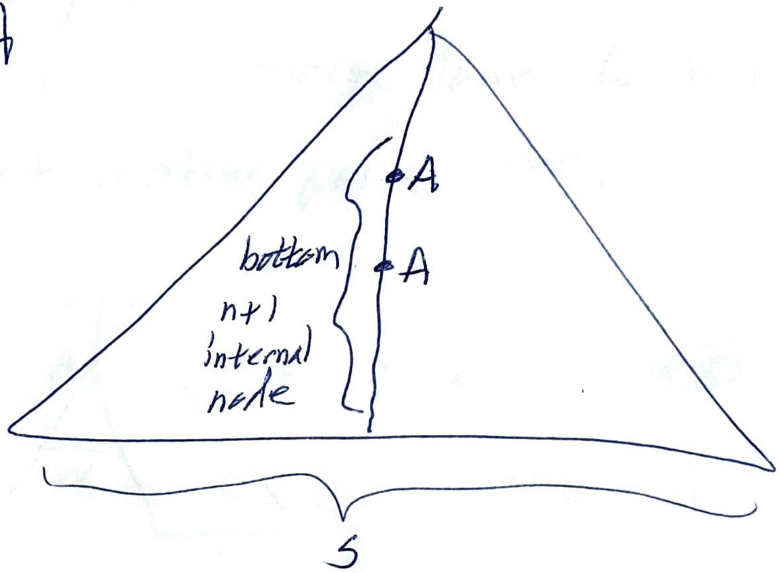
Let $n := |V|$

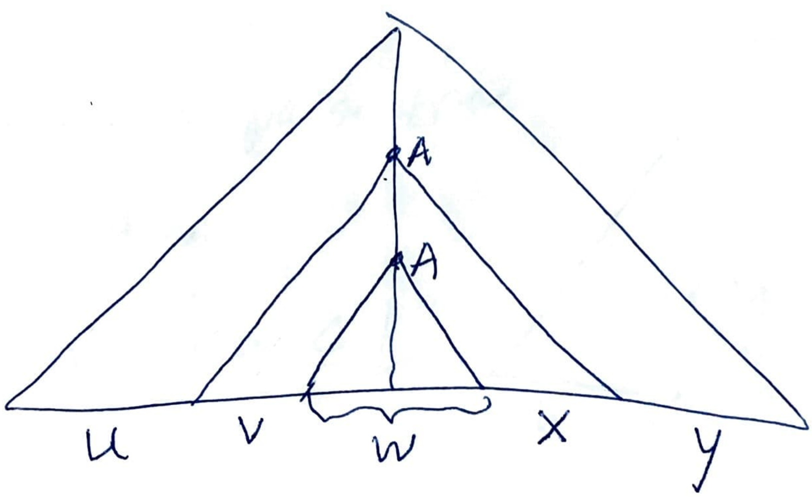
Let $d := \max\{ |\alpha| : A \to \alpha$ is a production in P for some $A \in V \}$

Let $p := d^{n+1}$. Let $s \in L$ be such that $|s| \geq p = d^{n+1}$. Let T be a minimum-size parse tree yielding s.



(5) Since $|s| \ge p = d^{n+1}$, T must have depth $\ge n+1$. So there is a path in T with $\ge n+1$ many internal nodes. Look at the $n+1$ many internal nodes lowest in the path. Since there are only n nonterminals, two of these internal nodes are labeled with the same nonterminal, say A





Let w be the descendants of the lower A, v, x be the descendants of the upper A to the left & right of w, respectively.
Let u, y be the rest of the leaves, to the left & right, respectively.

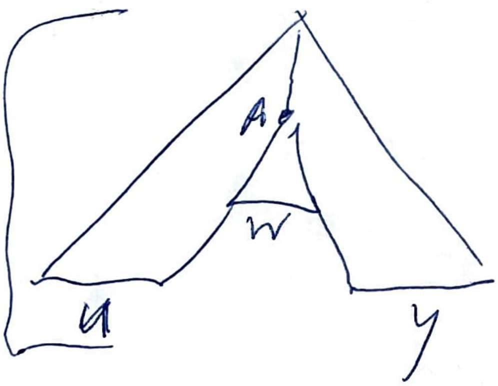
⑥ (a) $s = uvwxy$ ✓

(b) $|vwx| \le d^{n+1} = p$ because vwx form the descendants of the upper A (with height $\le n+1$)

(c) $\underline{|vx| > 0}$ (later) ← Otherwise $s = uvwxy = uwy$ ∴ smaller parse tree for s ↯

Choose any i

Spse $i = 0$: merge lower A with upper A to get another parse tree:



∴ $uwy = uv^0wx^0y \in L$

If $i > 1$: Get a parse tree yielding $uv^iwx^iy \in L$